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ON DYNAMIC STABILITY OF AN UNIPERIODIC MEDIUM THICKNESS PLATE BAND

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The aim of this contribution is to apply equations derived by Baron (2002) to the analysis of the dynamic stability of a periodically ribbed simply supported plate band. The general equation of motion for the plate band subjected to the time-dependent axial force was obtained. The considerations are related to the dynamic analysis for arbitrary boundary conditions. The obtained frequency equation can be treated as a certain generalization of the known Mathieu equation. By applying the procedure used for investigation of the Mathieu equation, two fundamental regions of the dynamic instability are determined. The obtained results are similar to those derived from known solutions, but also depend on the period l. We also deal with a new higher free vibration frequency.

Key words: modelling, dynamic stability, medium thickness plate, periodic structure

1. Introduction

The main aim of this contribution is to apply general equations of uniperiodic plates formulated by Baron (2002) to detect the effect of repetitive cell size on the dynamic plate behaviour in the case of dynamic instability. In the aforementioned paper a new modelling approach to the medium thickness elastic periodic plates was presented in the framework of the Hencky-Bolle theory. A new averaged 2D-model of Hencky-Bolle elastic plates with a one-directional periodic (uniperiodic) structure was obtained by using the tolerance averaging of partial differential equations with periodic coefficients, cf. Woźniak and Wierzbicki (2000). This model describes the influence of the repetitive (periodic) cell size on the overall plate behaviour (the microstructure length-scale effect).

In the present contribution we analyse the problem of dynamic plate instability related to the parametric resonance of the plate band, cf. Bolotin (1956). The plate band under consideration is simply supported on the opposite edges and subjected to a time-dependent compressive axial force acting in the direction normal to the edges, see Fig. 1. Hence, the governing equations of the medium thickness uniperiodic plate derived by Baron (2002) will be specified for a plate band made of an orthotropic homogeneous material and having a periodically variable thickness in the x_1-axis direction and constant in the x_2-axis direction, see Fig. 1. On these assumptions the frequency equation for the uniperiodic plate band will be derived by the tolerance averaging of the 2D-plate equations. Assuming that the compressive axial force is constant, the free vibration frequencies and static critical force will be calculated. For the time-dependent compressive force given by $N(t) = N_0 + N_1 \cos(pt)$, it will be shown that instead of the known Mathieu equation we obtain a forth-order differential equation. By applying the procedure similar to that used for investigations of the Mathieu equation, two fundamental dynamic instability regions are determined. The dynamic stability of plates periodic in two dimensions was investigated by Wierzbicki and Woźniak (2002).

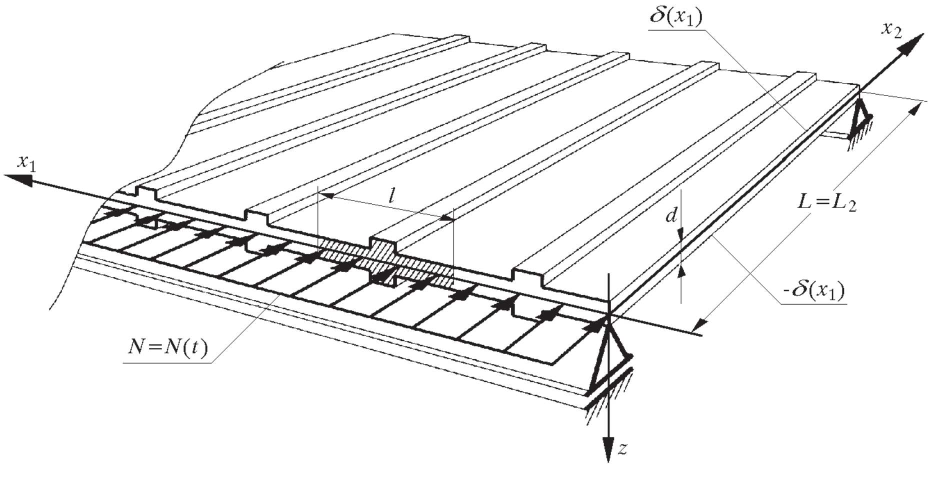

Fig. 1. Uniperiodically ribbed plate band

The obtained solutions related to the dynamic stability are similar to the known solutions but also depend on the periodic cell size, and hence they constitute a certain generalization of the results derived from the Mathieu equation. The results derived on the basis of the tolerance averaging model are compared to those derived in the framework of the asymptotic model, cf. Baron (2002).

Throughout the paper the subscripts $\alpha, \beta, ...$ run over $1, 2$, subscripts $i, j, ...$ over $1, 2, 3$ and superscripts $A, B, ...$ over $1, 2, ..., N$; summation convention holds for all aforementioned indices.

2. Preliminaries

To make the paper self-consistent we recall some fundamental concepts and denotations following Baron (2002).

By $\boldsymbol{x} = (x_1, x_2)$ we denote the Cartesian orthogonal coordinates of a point on the plate midplane $\Pi = (0, L_1) \times (0, L_2)$ and by z the Cartesian coordinate in the direction normal to the midplane. By $z = \pm\delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Pi$, we denote functions representing the upper and lower plate boundary, respectively; hence $2\delta(\boldsymbol{x})$ is the plate thickness at a point $\boldsymbol{x} \in \Pi$. By $\rho = \rho(\boldsymbol{x}, z)$ and $A_{ijkl}(\boldsymbol{x}, z)$ we denote the mass density and elastic moduli tensor of the plate, and assume that every plane $z = \text{const}$ is a plane of elastic symmetry. We also define $C_{\alpha\beta\gamma\delta} := A_{\alpha\beta\gamma\delta} - A_{\alpha\beta 33}A_{33\gamma\delta}(A_{3333})^{-1}$, $B_{\alpha\beta} := A_{\alpha 3\beta 3}$. We assume that $\delta(\cdot)$, $\rho(\cdot)$, $A_{ijkl}(\cdot)$ are periodic functions with the period l with respect to the x_1-coordinate, $\rho(\cdot)$, $A_{ijkl}(\cdot)$ are even functions of z, and all aforementioned functions are sufficiently regular ones of the x_2-coordinate. We also assume that $l \gg \max\delta(\boldsymbol{x})$. Let p^+ and p^- denote tractions (along the z-axis) situated on the upper and lower plate boundaries, respectively, $N_{\alpha\beta}$ be the prestressing tensor of the plate midplane and b be the z-coordinate of a constant body force acting along the z-axis direction. Furthermore, let t be the time coordinate.

The averaged value of an arbitrary integrable function $\varphi(x_1, x_2, t)$ in the periodicity interval $(x_1 - l/2, x_1 + l/2)$ will be denoted by

$$\langle\varphi\rangle(\boldsymbol{x}, t) = \frac{1}{l} \int\limits_{x_1 - l/2}^{x_1 + l/2} \varphi(\xi, x_2, t) \, d\xi \qquad \boldsymbol{x} = (x_1, x_2)$$

In the special case when $\varphi(\cdot)$ is an uniperiodic function, i.e. $\varphi(\cdot)$ is periodic only in the x_1-direction, the above averaged value is independent of x_1, and will be denoted by $\langle\varphi\rangle$.

Under denotations (in the formulae for $D_{\alpha\beta}$ there is no summation over α and β!)

$$\mu := \int_{-\delta}^{\delta} \rho\, dz \qquad J := \int_{-\delta}^{\delta} z^2\rho\, dz \qquad p := p^+ + p^- + b\langle\mu\rangle$$

$$G_{\alpha\beta\gamma\delta} := \int_{-\delta}^{\delta} z^2 C_{\alpha\beta\gamma\delta}\, dz \qquad D_{\alpha\beta} := \int_{-\delta}^{\delta} K_{\alpha\beta}B_{\alpha\beta}\, dz$$

where $K_{\alpha\beta}$ are the shear coefficients, we recall the system of equations for unknown displacements w and rotations ϑ_α

$$\begin{aligned}
&(G_{\alpha\beta\gamma\delta}\vartheta_{(\gamma,\delta)})_{,\beta} - D_{\alpha\beta}\vartheta_\beta - J\ddot{\vartheta}_\alpha = 0\\
&N^o_{\alpha\beta}w_{,\alpha\beta} + [D_{\alpha\beta}(\vartheta_\beta + w_{,\beta})]_{,\alpha} - \mu\ddot{w} + p = 0
\end{aligned}$$

representing the Hencky-Bolle theory of the medium thickness plates, cf. Jemielita (2001). For the periodic plates under consideration this is a system of equations with functional coefficients depending on x_1, which are highly oscillating and can be non-continuous. Direct solutions to the boundary value problems related to these equations are very complicated. That is why, some simplified models of the plates have been proposed, cf. Baron (2002). We can mention here homogenized models based on the asymptotic approach which leads to differential equations with constant coefficients. However, the asymptotic homogenization neglects the effect of periodicity cell size on the macrodynamic plate behaviour (length-scale effect). The new modelling approach, which describes the above length-scale effect was presented by Baron (2002). This approach applies the tolerance averaging technique to the plate equations, cf. Woźniak and Wierzbicki (2000). According to the tolerance averaging technique, we introduce the decompositions

$$\vartheta_\alpha = \vartheta^o_\alpha + \vartheta^*_\alpha \qquad\qquad w = w^o + w^*$$

where the *slowly varying functions* $w^o(\cdot)$, $\vartheta^o_\alpha(\cdot)$ are the averaged displacements and rotations, respectively, and the *oscillating functions* $w^*(\cdot)$, $\vartheta^*_\alpha(\cdot)$ are fluctuations of these fields, cf. Woźniak and Wierzbicki (2000). Using the procedure proposed by Baron (2002), the functions $w^*(\cdot)$, $\vartheta^*_\alpha(\cdot)$ can be approximated by means of the formulae

$$\begin{aligned}
&\vartheta^*_{x\alpha}(\boldsymbol{x},t) \cong h^a(x_1)\Theta^a_\alpha(\boldsymbol{x},t) \qquad && a = 1,2,...,n\\
&w^*_x(\boldsymbol{x},t) \cong g^A(x_1)W^A(\boldsymbol{x},t) \qquad && A = 1,2,...,N
\end{aligned} \tag{2.1}$$

where $\Theta^a_\alpha(\boldsymbol{x},t)$, $W^A(\boldsymbol{x},t)$ are new unknown functions which are slowly varying with respect to the x_1-coordinate, and $h^a(x_1)$, $g^A(x_1)$ are certain linear independent mode-shape functions satisfying the conditions

$$\begin{array}{lll} \langle Jh^a\rangle = 0 & \langle \mu g^A\rangle = 0 & h^a(x_1) \in O(l) \\ g^A(x_1) \in O(l) & lh^a_{,1}(x_1),\ lg^A_{,1}(x_1) \in O(l) & \end{array}$$

Taking into account the aforementioned conditions we shall also introduce the functions

$$\overline{h}^a = l^{-1}h^a \qquad\qquad \overline{g}^A = l^{-1}g^A$$

which are of order $O(1)$ when $l \to 0$. In most cases, the mode-shape functions are postulated *a priori*, and hence formulae (2.1) can be interpreted as a certain kinematic hypothesis related to the form of expected displacement disturbances caused by the plate periodic structure. The degree of approximation $\cong$ in (2.1) depends on the number of terms on the right-hand sides in (2.1).

The governing equations of the averaged 2D-model of nonhomogeneous, medium thickness elastic plates with a one-directional periodic structure, which have been derived by Baron (2002), consist of:

— equations of motion

$$\begin{aligned} &M_{\alpha\beta,\beta} - Q_\alpha - \langle J\rangle \ddot{\vartheta}^o_\alpha = 0 \\ &N^o_{\alpha\beta} w^o_{,\alpha\beta} + Q_{\alpha,\alpha} - \langle \mu\rangle \ddot{w}^o + p = 0 \end{aligned} \tag{2.2}$$

— equations for Θ^a, W^A

$$\begin{aligned} &l^2\langle J\overline{h}^a\overline{h}^b\rangle \ddot{\Theta}^b_\alpha + M^a_\alpha - \widetilde{M}^a_{\alpha,2} = 0 \\ &l^2\langle \mu\overline{g}^A\overline{g}^B\rangle \ddot{W}^B + Q^A - l\widetilde{Q}^A_{,2} - N^o_{\alpha\beta}\langle \overline{g}^A\rangle w^o_{,\alpha\beta} - l\langle \overline{g}^A p\rangle = 0 \end{aligned} \tag{2.3}$$

— constitutive equation

$$\begin{aligned} M_{\alpha\beta} &= \langle G_{\alpha\beta\gamma\delta}\rangle \vartheta^o_{(\gamma,\delta)} + \langle h^a_{,1} G_{\alpha\beta1\delta}\rangle \Theta^a_\delta + l\langle \overline{h}^a G_{\alpha\beta2\delta}\rangle \Theta^a_{\delta,2} \\ Q_\alpha &= \langle D_{\alpha\beta}\rangle(\vartheta^o_\beta + w^o_{,\beta}) + l\langle \overline{h}^a D_{\alpha\beta}\rangle \Theta^a_\beta + \langle g^A_{,1} D_{\alpha1}\rangle W^A + l\langle \overline{g}^A D_{\alpha2}\rangle W^A_{,2} \\ M^a_\alpha &= \langle h^a_{,1} h^b_{,1} G_{\alpha11\delta}\rangle \Theta^b_\delta + \langle h_{,1} G_{\alpha1\gamma\delta}\rangle \vartheta^o_{(\gamma,\delta)} + l\langle h^a_{,1}\overline{h}^b G_{\alpha12\delta}\rangle \Theta^b_{\delta,2} + \\ &\quad + l^2\langle \overline{h}^a\overline{h}^b D_{\alpha\beta}\rangle \Theta^b_\beta + l\langle \overline{h}^a D_{\alpha\beta}\rangle(\vartheta^o_\beta + w^o_{,\beta}) + \\ &\quad + l\langle \overline{h}^a g^A_{,1} D_{\alpha1}\rangle W^A + l^2\langle \overline{h}^a\overline{g}^A D_{\alpha2}\rangle W^A_{,2} \end{aligned} \tag{2.4}$$

$$
\begin{aligned}
\widetilde{M}^a_\alpha &= \langle \overline{h}^a h^b_{,1} G_{\alpha 21\delta}\rangle \Theta^b_\delta + \langle \overline{h}^a G_{\alpha 2\gamma\delta}\rangle \vartheta^o_{(\gamma,\delta)} + l\langle \overline{h}^a \overline{h}^b G_{\alpha 22\delta}\rangle \Theta^b_{\delta,2} \\
Q^A &= \langle g^A_{,1} g^B_{,1} D_{11}\rangle W^B + \langle g^A_{,1} D_{1\beta}\rangle (\vartheta^o_\beta + w^o_{,\beta}) + l\langle g^A_{,1} \overline{h}^a D_{1\beta}\rangle \Theta^a_\beta + \\
&\quad + l\langle g^A_{,1} \overline{g}^B D_{12}\rangle W^B_{,2} \\
\widetilde{Q}^A &= \langle \overline{g}^A g^B_{,1} D_{21}\rangle W^B + \langle \overline{g}^A D_{2\beta}\rangle (\vartheta^o_\beta + w^o_{,\beta}) + l\langle \overline{g}^A h^a D_{2\beta}\rangle \Theta^a_\beta + \\
&\quad + l\langle \overline{g}^A \overline{g}^B D_{22}\rangle W^B_{,2}
\end{aligned}
$$

Equations (2.2)-(2.4) constitute the starting point for the analysis of parametric vibrations and dynamic stability of the medium thickness periodically ribbed plate band. In a general case, the coefficients in (2.2)-(2.4) can depend on the x_2-coordinate.

3. Equations of the uniperiodic plate band

In this section we specify equations (2.2)-(2.4) for the plate band made of an orthotropic homogeneous material, having the uniperiodic (with the period l) variable thickness $\delta(x_1)$ along the x_1-axis, see Fig. 1. In this case, the coefficients in (2.2)-(2.4) are constant. Taking into account the orthotropy of the plate we denote

$$
\begin{aligned}
&G_{11} = G_{1111} && G_{22} = G_{2222} \\
&G_{12} = G_{1122} = G_{2211} && G = G_{1212} = G_{1221} = G_{2112} = G_{2121} \\
&D_1 = D_{11} && D_2 = D_{22}
\end{aligned}
$$

In the subsequent considerations only two mode-shape functions will be taken into account. Hence, we define

$$
h = h^1(x_1) = l\overline{h}(x_1) \qquad\qquad g = g^1(x_1) = l\overline{g}(x_1)
$$

where the values of $\overline{h}(x_1)$ and $\overline{g}(x_1)$ are of order $O(1)$ when $l \to 0$. We also denote

$$
\begin{aligned}
&\Theta_1(\boldsymbol{x},t) = \Theta^1_1(\boldsymbol{x},t) && \Theta_2(\boldsymbol{x},t) = \Theta^1_2(\boldsymbol{x},t) && W(\boldsymbol{x},t) = W^1(\boldsymbol{x},t) \\
&\vartheta_1(\boldsymbol{x},t) = \vartheta^o_1(\boldsymbol{x},t) && \vartheta_2(\boldsymbol{x},t) = \vartheta^o_2(\boldsymbol{x},t) && w(\boldsymbol{x},t) = w^o(\boldsymbol{x},t)
\end{aligned}
$$

For the cylindrical bending of the plate band in the Ox_2z-plane, all unknowns are independent on x_1. Setting $x = x_2$, equations (2.2)-(2.4) reduce to the form

$$
\begin{aligned}
&\langle G\rangle\vartheta_1'' - \langle D_1\rangle\vartheta_1 - \langle J\rangle\ddot{\vartheta}_1 + l\langle\overline{h}D_1\rangle\Theta_1 + \langle h_{,1}G\rangle\Theta_{21}' - \langle g_{,1}D_1\rangle W = 0\\
&\langle G_{22}\rangle\vartheta_2'' - \langle D_2\rangle\vartheta_2 - \langle J\rangle\ddot{\vartheta}_2 + \langle h_{,1}G_{12}\rangle\Theta_1' - l\langle\overline{h}D_2\rangle\Theta_2 - \langle D_2\rangle w' = 0\\
&(N_{22} + \langle D_2\rangle)w'' - \langle\mu\rangle\ddot{w} + \langle D_2\rangle\vartheta_2' + l\langle\overline{h}D_2\rangle\Theta_2' + p = 0\\
&-l^2\langle\overline{h}^2G\rangle\Theta_1'' + l^2\langle\overline{h}^2D_1\rangle\Theta_1 + \langle h_{,1}^2G_{11}\rangle\Theta_1 + l^2\langle\overline{h}^2J\rangle\ddot{\Theta}_1 +\\
&\qquad + l^2\langle\overline{h}D_1\rangle\vartheta_1 + \langle h_{,1}G_{12}\rangle\vartheta_2' + l\langle\overline{h}g_{,1}D_1\rangle W = 0 \qquad\qquad (3.1)\\
&-l^2\langle\overline{h}^2G_{22}\rangle\Theta_2'' + (l^2\langle\overline{h}^2D_2\rangle + \langle h_{,1}^2G\rangle)\Theta_2 + l^2\langle\overline{h}^2J\rangle\ddot{\Theta}_2 +\\
&\qquad + \langle h_{,1}G\rangle\vartheta_1' + l\langle\overline{h}D_2\rangle(\vartheta_2 + w') + l^2\langle\overline{h}\overline{g}D_2\rangle W' = 0\\
&-l^2\langle\overline{g}^2D_2\rangle W'' + \langle g_{,1}^2D_1\rangle W + l^2\langle\overline{g}^2\mu\rangle\ddot{W} + \langle g_{,1}D_1\rangle\vartheta_1 +\\
&\qquad + l\langle\overline{h}g_{,1}D_1\rangle\Theta_1 - l^2\langle\overline{g}\overline{h}D_2\rangle\Theta_2' = 0
\end{aligned}
$$

where the denotations $\partial f/\partial x = f'(x)$, $\partial^2 f/\partial x^2 = f''(x)$ for an arbitrary differentiable function f are introduced. Thus, the problem under consideration is governed by the system of six partial differential equations (3.1) for six unknown functions: the averaged rotations $\vartheta_\alpha(x,t)$, averaged displacement $w(x,t)$ and extra unknowns $\Theta_\alpha(x,t)$, $W(x,t)$. Equations (3.1) have a physical sense provided that $\Theta_\alpha(\cdot,t)$ and $W(\cdot,t)$ are slowly varying functions, cf. Woźniak and Wierzbicki (2000).

Taking into account the symmetric form of the periodic cell, cf. Fig. 1, we assume that the mode-shape function $h(x_1)$ is odd while the mode-shape function $g(x_1)$ is even. In this case, the system of equations (3.1) can be separated into two independent systems. The first of these systems is the following system of equations describing evolutions of the functions $\vartheta_1(\boldsymbol{x},t)$ and $\Theta_2(\boldsymbol{x},t)$

$$
\begin{aligned}
&\langle G\rangle\vartheta_1'' - \langle D_1\rangle\vartheta_1 - \langle J\rangle\ddot{\vartheta}_1 + \langle h_{,1}G\rangle\Theta_2' = 0\\
&-l^2\langle\overline{h}^2G_{22}\rangle\Theta_1'' + (l^2\langle\overline{h}^2D_1\rangle + \langle h_{,1}^2G\rangle)\Theta_2 + l^2\langle\overline{h}^2J\rangle\ddot{\Theta}_2 + \langle h_{,1}G\rangle\vartheta_1' = 0
\end{aligned}
\qquad (3.2)
$$

The second one describes evolutions of the functions $\vartheta_2(\boldsymbol{x},t)$, $w(\boldsymbol{x},t)$, $\Theta_1(\boldsymbol{x},t)$, $W(\boldsymbol{x},t)$

$$
\begin{aligned}
&\langle G_{22}\rangle\vartheta_2'' - \langle D_2\rangle\vartheta_2 - \langle J\rangle\ddot{\vartheta}_2 + \langle h_{,1}G_{12}\rangle\Theta_1' - \langle D_2\rangle w' = 0\\
&(N_{22} + \langle D_2\rangle)w'' - \langle\mu\rangle\ddot{w} + \langle D_2\rangle\vartheta_2' + p = 0\\
&-l^2\langle\overline{h}^2G\rangle\Theta_1'' + (l^2\langle\overline{h}^2D_1\rangle + \langle h_{,1}^2G_{11}\rangle)\Theta_1 + l^2\langle\overline{h}^2J\rangle\ddot{\Theta}_1 + \\
&\qquad +\langle h_{,1}G_{12}\rangle\vartheta_2' + l\langle\overline{h}g_{,1}D_1\rangle W = 0\\
&-l^2\langle\overline{g}^2D_2\rangle W'' + \langle g_{,1}^2D_1\rangle W + l^2\langle\overline{g}^2\mu\rangle\ddot{W} + l\langle\overline{h}g_{,1}D_1\rangle\Theta_1 = 0
\end{aligned}
\tag{3.3}
$$

Neglecting the rotational inertia terms and assuming the homogeneous initial conditions, equations (3.2) yields $\vartheta_1 = 0$ and $\Theta_2 = 0$. Bearing in mind that $N_{22} = -N$, $N = N(t)$, together with $p = p(\boldsymbol{x}, t)$, neglecting terms involving J and taking into account denotations

$$
\begin{aligned}
&\varphi_1[w] = [l^2\langle\overline{h}^2G\rangle\langle G_{22}\rangle w'' - (l^2\langle\overline{h}^2D_1\rangle + \langle h_{,1}^2G_{11}\rangle)H_o w]''\\
&\varphi_2[w] = [l^2\langle\overline{h}^2G\rangle\langle G_{22}\rangle w'' - \langle h_{,1}^2G_{11}\rangle H_1 w]''\\
&\psi_1[w] = \varphi_1[w] - \langle D_2\rangle[l^2\langle\overline{h}^2G\rangle w'' - (l^2\langle\overline{h}^2D_1\rangle + \langle h_{,1}^2G_{11}\rangle)w]\\
&\psi_2 = \varphi_2[w] - \langle D_2\rangle(l^2\langle\overline{h}^2G\rangle w'' - \langle h_{,1}^2G_{11}\rangle w)\\
&\zeta_1[p] = l^2\langle\overline{h}^2G\rangle(\langle G_{22}\rangle p'' - \langle D_2\rangle p)'' - (l^2\langle\overline{h}^2D_1\rangle + \langle h_{,1}^2G_{11}\rangle)(H_o p'' - \langle D_2\rangle p)\\
&\zeta_2[p] = l^2\langle\overline{h}^2G\rangle(\langle G_{22}\rangle p'' - \langle D_2\rangle p)'' - \langle h_{,1}^2G_{11}\rangle(H_1 p'' - \langle D_2\rangle p)
\end{aligned}
$$

where

$$
H_o = \langle G_{22}\rangle - \frac{\langle h_{,1}G_{12}\rangle^2}{l^2\langle\overline{h}^2D_1\rangle + \langle h_{,1}^2G_{11}\rangle} \qquad\qquad H_1 = \langle G_{22}\rangle - \frac{\langle h_{,1}G_{12}\rangle^2}{\langle h_{,1}^2G_{11}\rangle}
$$

we obtain from equations (3.3) a single equation for the averaged deflection $w(x, t)$ in the form

$$
\begin{aligned}
&-N(l^2\langle\overline{g}^2D_2\rangle\psi_1'' - \langle g_{,1}^2D_1\rangle\psi_2)'' + \langle D_2\rangle(l^2\langle\overline{g}^2D_2\rangle\varphi_1'' - \langle g_{,1}^2D_1\rangle\varphi_2)'' -\\
&-l^2\langle\overline{g}^2\mu\rangle(-\ddot{N}\psi_1 + \langle D_2\rangle\ddot{\varphi}_1 + N\ddot{\psi}_1)'' - \langle\mu\rangle(l^2\langle\overline{g}^2D_2\rangle\ddot{\psi}_1'' - \langle g_{,1}^2D_1\rangle\ddot{\psi}_2) +\\
&+l^2\langle\overline{g}^2\mu\rangle\frac{\partial^4}{\partial t^4}\psi_1 = l^2\langle\overline{g}^2D_2\rangle\zeta_1'' - \langle g_{,1}^2D_1\rangle\zeta_2 - l^2\langle\overline{g}^2\mu\rangle\ddot{\zeta}_1
\end{aligned}
\tag{3.4}
$$

Equation (3.4) will be treated as a general equation of motion of an uniperiodic, medium thickness plate band.

4. Frequency equation

By separating the variables, the unknown function $w(x,t)$ in (3.4) will be assumed in the form

$$w(x,t) = w_o(x)T(t) \tag{4.1}$$

Let use assume that the plate band is simply supported on the edges $x = 0$, $x = L$. In this case, the unknown function $w_o(x)$ will be assumed in the form

$$w_o(x) = \sum_{n=1}^{\infty} w_n \sin(k_n x) \qquad k_n = \frac{n\pi}{L} \qquad n = 1, 2, ... \tag{4.2}$$

In the special case, when $p = 0$, taking into account (4.1), (4.2), and denoting

$$\begin{aligned}
H_n &= \langle G_{22}\rangle\Big(1 + k_n^2 l^2 \frac{\langle \overline{h}^2 G\rangle}{l^2\langle \overline{h}^2 D_1\rangle + \langle h_{,1}^2 G_{11}\rangle}\Big) - \frac{\langle h_{,1} G_{12}\rangle^2}{l^2\langle \overline{h}^2 D_1\rangle + \langle h_{,1}^2 G_{11}\rangle} \\
D_n &= \langle D_2\rangle\Big(1 + k_n^2 l^2 \frac{\langle \overline{h}^2 G\rangle}{l^2\langle \overline{h}^2 D_1\rangle + \langle h_{,1}^2 G_{11}\rangle}\Big) \\
\kappa_n &= \frac{\langle D_2\rangle}{k_n^2 H_n + D_n} \qquad\qquad \mathring{\kappa} = \Big(1 + \frac{\langle h_{,1}^2 G_{11}\rangle}{l^2\langle \overline{h}^2 D_1\rangle}\Big)^{-1} \\
c_1 &= 1 - \kappa_n \mathring{\kappa} \frac{\langle G_{22}\rangle}{H_n} + k_n^2 l^2 \frac{\langle \overline{g}^2 D_2\rangle}{\langle g_{,1}^2 D_1\rangle} \\
c_2 &= 1 - \kappa_n \mathring{\kappa} \Big(1 + \frac{k_n^2 \langle G_{22}\rangle}{\langle D_2\rangle}\Big) + k_n^2 l^2 \frac{\langle \overline{g}^2 D_2\rangle}{\langle g_{,1}^2 D_1\rangle}
\end{aligned} \tag{4.3}$$

we obtain from (3.4) the following frequency equation for the simply supported uniperiodic plate band

$$\begin{aligned}
& l^2 \frac{d^4 T}{dt^4} + \Big[\frac{c_2\langle g_{,1}^2 D_1\rangle}{\langle \overline{g}^2 \mu\rangle} + \frac{k_n^2 l^2}{\langle \mu\rangle}(-N + k_n^2 H_n \kappa_n)\Big]\frac{d^2 T}{dt^2} + \\
& + k_n^2 \Big[\frac{\langle g_{,1}^2 D_1\rangle}{\langle \overline{g}^2 \mu\rangle\langle \mu\rangle}(-c_2 N + c_1 k_n^2 H_n \kappa_n) - l^2 \frac{d^2 N}{dt^2}\frac{1}{\langle \mu\rangle}\Big] T = 0
\end{aligned} \tag{4.4}$$

If the plate satisfy the condition $d \ll l \ll L$, where $d = 2\max\delta(\boldsymbol{x})$, we obtain $c_2 \approx c_1$, $\langle D_n\rangle \approx \langle D_2\rangle$ and then

$$H_n \approx H = \langle G_{22}\rangle - \frac{\langle h_{,1} G_{12}\rangle^2}{l^2\langle \overline{h}^2 D_1\rangle + \langle h_{,1}^2 G_{11}\rangle} \qquad\qquad \kappa_n \approx \frac{\langle D_2\rangle}{k_n^2 H + \langle D_2\rangle} \tag{4.5}$$

Taking into account (4.5) and assuming that the axial force is compressive ($N < 0$) and constant, we obtain from (4.4) the frequency equation

$$\begin{aligned} & l^2\frac{d^4T}{dt^4} + \Big[\frac{c_2\langle g_{,1}^2 D_1\rangle}{\langle\overline{g}^2\mu\rangle} + \frac{k_n^2 l^2}{\langle\mu\rangle}(-N + k_n^2 H_n\kappa_n)\Big]\frac{d^2T}{dt^2} + \\ & +k_n^2\frac{c_2\langle g_{,1}^2 D_1\rangle}{\langle\overline{g}^2\mu\rangle\langle\mu\rangle}(-N + k_n^2 H\kappa_n)T = 0 \end{aligned} \tag{4.6}$$

From the above equation, after some transformations, we derive the following formulae for the higher ω_1 and lower ω_2 free vibration frequencies

$$\omega_1^2 = \frac{c_2\langle g_{,1}^2 D_1\rangle}{l^2\langle\overline{g}^2\mu\rangle} \qquad\qquad \omega_2^2 = \frac{k_n^2}{\langle\mu\rangle}(-N + k_n^2 H\kappa_n) \tag{4.7}$$

For the static critical force we obtain

$$N_{kr,n} = k_n^2 H\kappa_n \tag{4.8}$$

It must be emphasised that in the first approximation the lower frequency ω_2 and the static critical force $N_{kr,n}$ coincide with those derived from the asymptotic model which will be presented in Section 6. The higher frequency ω_1 will be obtained by using the tolerance averaging model.

5. Dynamic stability

In this section we shall investigate the dynamic instability for the uniperiodic, simply supported plate band, described by equation (3.4). We shall examine the case of dynamic instability caused by the parametric resonance, cf. Bolotin (1956). Using the standard procedure, we assume that the compressive axial force in the plate midplane is time dependent and governed by the relation

$$N(t) = N_0 + N_1\cos(pt)$$

where N_0, N_1 are constant. The solution to equation (3.4) will be assumed in the form

$$w(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(k_n x) \qquad\qquad k_n = \frac{n\pi}{L} \qquad\qquad n = 1,2,... \tag{5.1}$$

Now, the nth free vibration frequency have the form

$$\Omega_n^2 = \frac{k_n^4 H\kappa_n}{\langle\mu\rangle}\Big(1 - \frac{N_0}{N_{kr,n}}\Big)$$

At the same time, the modulation factor (depth of the modulation) has the form

$$2\overline{\mu}_n = \frac{N_1}{N_{kr,n} - N_0}\Big(1 - \frac{p^2}{\omega_1^2}\Big)$$

Taking into account denotations (4.3) and assumptions (4.5), we finally obtain

$$l^2\frac{d^4T_n}{dt^4} + \frac{c_2\langle g_{,1}^2 D_1\rangle}{\langle\overline{g}^2\mu\rangle}\Big\{\frac{d^2T_n}{dt^2} + \Omega_n^2[1 - 2\overline{\mu}_n\cos(pt)]T_n\Big\} = 0 \tag{5.2}$$

It must be emphasised that equation (5.2) is a certain generalization of the known Mathieu equation; it takes the form of the Mathieu equation provided that in (5.2) the length-scale effect is neglected.

The analysis of dynamic stability leads to the determination of the instability regions on the $(p/\Omega_n, \overline{\mu}_n)$-plane. Hence, we ask whether, for the given quotient of the exciting frequency of the axial force p and the free vibration frequency Ω_n and for the given modulation factor $\overline{\mu}_n$, the plate band vibrations are stable or instable. Thus, we have to determine the instability regions (resonance regions) for solutions to equation (5.2). Within the resonance regions vibrations grow up in an unlimited way as $t \to \infty$. Outside and at the boundaries of the resonance regions there exist periodic solutions to equation (5.2) with the parametric excitation periods $\overline{T}_p = 2\pi/p$ and $2\overline{T}_p$. The considerations will be restricted to the parametric vibrations for the first harmonic components of series (5.1). The subsequent resonance regions for $3\overline{T}_p$, $4\overline{T}_p, \dots$ and for higher harmonic components $(n = 2, 3, \dots)$ do not play an important role in most engineering problems. That is why we are looking for the instability solutions for the equation

$$\frac{d^4T}{dt^4} + \omega^2\Big[\frac{d^2T}{dt^2} + \Omega^2\Big(1 - 2\overline{\mu}\cos(pt)\Big)T\Big] = 0 \tag{5.3}$$

where, for the sake of simplicity, we neglected the index 1 and denoted

$$\omega^2 = \omega_1^2 = \frac{c_2\langle g_{,1}^2 D_1\rangle}{l^2\langle\overline{g}^2\mu\rangle}$$

Looking for the solutions with the period $2\overline{T}_p$ related to the boundaries of the first instability region we substitute

$$T(t) = \sum_{i=1,3,\dots}^{\infty}\Big(a_i\sin\frac{ipt}{2} + b_i\cos\frac{ipt}{2}\Big)$$

into (5.3). After comparing the coefficients of pertinent trigonometric functions to zero, we obtain the following infinite system of the linear algebraic equations

$$\begin{aligned}&\Big[1+\overline{\mu}-\Big(\frac{p}{2\Omega}\Big)^2+\varepsilon^2\Big(\frac{p}{2\Omega}\Big)^4\Big]a_1-\overline{\mu}a_3=0\\&\Big[1-\Big(\frac{ip}{2\Omega}\Big)^2+\varepsilon^2\Big(\frac{ip}{2\Omega}\Big)^4\Big]a_i-\overline{\mu}(a_{i-2}+a_{i+2})=0\qquad i=3,5,...\end{aligned} \tag{5.4}$$

$$\begin{aligned}&\Big[1-\overline{\mu}-\Big(\frac{p}{2\Omega}\Big)^2+\varepsilon^2\Big(\frac{p}{2\Omega}\Big)^4\Big]b_1-\overline{\mu}b_3=0\\&\Big[1-\Big(\frac{ip}{2\Omega}\Big)^2+\varepsilon^2\Big(\frac{ip}{2\Omega}\Big)^4\Big]b_i-\overline{\mu}(b_{i-2}+b_{i+2})=0\qquad i=3,5,...\end{aligned} \tag{5.5}$$

where $\varepsilon=\Omega/\omega$. For sufficiently small values of the modulation factor, the characteristic determinants of systems (5.4) and (5.5) can be approximated by the first components of relations (5.4)1 and $(5.5)_1$. In this case, in the $(p/\Omega,\overline{\mu})$-plane, we obtain the boundaries of the first instability region given by

$$\Big(\frac{p}{\Omega}\Big)^2\approx 1+\overline{\mu}\qquad\qquad \Big(\frac{p}{\Omega}\Big)^2\approx 1-\overline{\mu} \tag{5.6}$$

and an extra condition for the excitation force frequency $p\neq 2\omega$.

Solutions (5.6) are similar to the known solutions, Bolotin (1956). However, the free vibration frequency Ω and the modulation factor $\overline{\mu}$ depend on the period l. We have also obtained the extra resonance frequency for the axial excitation force.

Analogously, we look for the solution with the period $\overline{T}_p$. Setting

$$F(t)=b_o+\sum_{i=2,4,...}^{\infty}\Big(a_i\sin\frac{ipt}{2}+b_i\cos\frac{ipt}{2}\Big)$$

we obtain two homogeneous, infinite systems of linear algebraic equations

$$\begin{aligned}&b_o-\overline{\mu}b_2=0\\&\Big[1-\Big(\frac{p}{\Omega}\Big)^2+\varepsilon^2\Big(\frac{p}{\Omega}\Big)^4\Big]b_2-\overline{\mu}(2b_o+b_4)=0\\&\Big[1-\Big(\frac{ip}{2\Omega}\Big)^2+\varepsilon^2\Big(\frac{ip}{2\Omega}\Big)^4\Big]b_i-\overline{\mu}(b_{i+2}+b_{i-2})=0\qquad i=4,6,...\end{aligned} \tag{5.7}$$

$$
\begin{aligned}
&\Big[1-\Big(\frac{p}{\Omega}\Big)^2+\varepsilon^2\Big(\frac{p}{\Omega}\Big)^4\Big]a_2-\overline{\mu}a_4=0 \\
&\Big[1-\Big(\frac{ip}{2\Omega}\Big)^2+\varepsilon^2\Big(\frac{ip}{2\Omega}\Big)^4\Big]a_i-\overline{\mu}(a_{i+2}+a_{i-2})=0 \qquad i=4,6,...
\end{aligned}
\tag{5.8}
$$

For small values of the modulation factor, the dimensions of the characteristic determinants related to systems (5.7) and (5.8) can be restricted to two rows and two columns. Hence, by applying to (5.7) and (5.8) a similar procedure to that in the analysis of the classical Mathieu equation, we obtain the boundary of the second instability region

$$
\begin{aligned}
&\Big(\frac{p}{\Omega}\Big)^2=\underline{1-2\overline{\mu}^2}+\varepsilon^2(1-2\overline{\mu}^2) \\
&\Big(\frac{p}{\Omega}\Big)^2\approx\underline{1+\frac{1}{3}\overline{\mu}^2}+\varepsilon^2\frac{9-14\overline{\mu}^2}{9+8\overline{\mu}^2}
\end{aligned}
\tag{5.9}
$$

and the extra resonance frequency for the axial excitation force $p \neq \omega$ and $p \neq 0.5\omega$. In (5.9) we underlined the terms which are qualitatively conformable to the known solutions, Bolotin (1956). The length-scale effect described by the terms including coefficients depending on ε, yields a correction of the boundaries of the instability regions.

The above method of analysis of the parametric resonance problem seems to be the simplest possible. It yields sufficiently good results provided that the modulation factor satisfies the condition $\overline{\mu} < 0.6$. The obtained relations confirm the correctness of the presented averaged 2D-model of a medium thickness uniperiodic elastic plate. By applying this model to the analysis of dynamic instability of the considered plate, we obtained a certain generalization of the classical results, which are being found when ε tends to zero.

6. Asymptotic model

In this section we recall the procedure presented in Sections 3-5 for equations of the asymptotic model derived by Baron (2002). We take into account the assumptions and denotations given in Section 3. In the framework of the asymptotic model, obtained for $l \to 0$, the following system of equations of the plate band can be derived

$$
\begin{aligned}
&\langle G\rangle\vartheta_1'' - \langle D_1\rangle\vartheta_1 - \langle J\rangle\ddot{\vartheta}_1 + \langle h_{,1}G\rangle\Theta_2' = 0\\
&\langle G_{22}\rangle\vartheta_2'' - \langle D_2\rangle\vartheta_2 - \langle J\rangle\ddot{\vartheta}_2 + \langle h_{,1}G_{12}\rangle\Theta_1' - \langle D_2\rangle W' = 0\\
&(N_{22} + \langle D_2\rangle)w'' - \langle\mu\rangle\ddot{w} + \langle D_2\rangle\vartheta_2' + p = 0\\
&\langle h_{,1}^2G_{11}\rangle\Theta_1 + \langle h_{,1}G_{12}\rangle\vartheta_2' = 0\\
&\langle h_{,1}^2G\rangle\Theta_2 + \langle h_{,1}G\rangle\vartheta_1' = 0\\
&\langle g_{,1}^2D_1\rangle W + \langle g_{,1}D_1\rangle\vartheta_1 = 0
\end{aligned}
\tag{6.1}
$$

On assumption that the mode-shape function $g(x_1)$ is even, after neglecting the rotational inertial terms, and for the homogeneous initial conditions, we obtain $\vartheta_1 = 0$, $\Theta_2 = 0$ and $W = 0$. In this case, introducing, as in the previous section, the denotations $N_{22} = -N$, $N = N(t)$ the system of equations (6.1) reduces to the following equation of motion

$$
[H_o w'' - N(H_o w'' - \langle D_2\rangle w)]'' - \langle\mu\rangle(H_o\ddot{w}'' - \langle D_2\rangle\ddot{w}) = H_o p'' - \langle D_2\rangle p \tag{6.2}
$$

where

$$
H_o = \langle G_{22}\rangle - \frac{\langle h_{,1}G_{12}\rangle^2}{\langle h_{,1}^2G_{11}\rangle}
$$

Assuming that the plate band is simply supported on the edges $x = 0$, $x = L$ and for $p = 0$, by separating the variables, we obtain from (6.2) the following frequency equation

$$
\frac{d^2T}{dt^2} + \frac{k_n^2}{\langle\mu\rangle}(-N + k_n^2 H_o\kappa_n)T = 0 \tag{6.3}
$$

where

$$
\kappa_n = \frac{\langle D_2\rangle}{k_n^2 H_o + \langle D_2\rangle} \qquad\qquad k_n = \frac{n\pi}{L} \qquad n = 1, 2, ...
$$

If the axial force N is constant and compressive, then from (6.3) we obtain the free vibration frequency given by

$$
\omega_o^2 = \frac{k_n^2}{\langle\mu\rangle}(-N + k_n^2 H_o\kappa_n)
$$

and the static critical force

$$
N_{kr,n} = k_n^2 H_o\kappa_n
$$

In the framework of the asymptotic model described above it is not possible to determine the structural (higher) free vibration frequency.

When investigating the plate dynamic stability (i.e. an instability caused by the parametric resonance), we assume that the compressive axial force is given by

$$N(t) = N_0 + N_1 \cos(pt)$$

If we look for a solution to (6.3), in the form (5.1), then we arrive at the known Mathieu equation

$$\frac{d^2 T_n}{dt^2} + \Omega_n^2 (1 - 2\overline{\mu} \cos(pt)) T_n = 0$$

where

$$2\overline{\mu}_n = \frac{N_1}{N_{kr,n} - N_0}$$

is a modulation factor and

$$\Omega_n^2 = \frac{k_n^2 H_o \kappa_n}{\langle \mu \rangle} \Big(1 - \frac{N_0}{N_{kr,n}}\Big)$$

is the nth free vibration frequency. Using the procedure for the determination of the instability region boundaries described in Section 5, we obtain:
— for the first instability region (vibrations with period $2\overline{T}_p$)

$$\Big(\frac{p}{2\Omega}\Big)^2 \approx 1 + \overline{\mu} \qquad \Big(\frac{p}{2\Omega}\Big)^2 \approx 1 - \overline{\mu}$$

— for the second instability region (vibrations with period $\overline{T}_p$)

$$\Big(\frac{p}{\Omega}\Big)^2 \approx 1 - 2\overline{\mu}^2 \qquad \Big(\frac{p}{\Omega}\Big)^2 \approx 1 + \frac{1}{3}\overline{\mu}^2$$

After comparing the results obtained from the tolerance averaging model with those derived from the asymptotic one, the following conclusions can be formulated:

- the square of the lower resonance frequency ω_o^2 calculated from the asymptotic model is an approximation of order $O(l^2)$ of the frequency ω_2^2 derived from the tolerance averaging model, i.e., $\omega_2^2 = \omega_o^2 + O(l^2)$;
- the higher resonance frequency ω_1^2, determined from equation $(4.7)_1$, cannot be derived from the tolerance averaging model;
- in the asymptotic model, the analysis of the dynamic stability of the plate band leads to the known Mathieu equation.

7. Conclusions

This contribution represents both a supplement and continuation of the paper by Baron (2002). By applying the tolerance averaging procedure formulated in the above paper, the general equation of a new averaged 2D-model of the medium thickness elastic plates with a one-directional periodic structure has been obtained. This model takes into account the effect of the period length on the overall plate behaviour.

The aim of this contribution was to apply the general equations of uniperiodic plates formulated by Baron (2002) for detection of the influence of the effect of repetitive cell size on the dynamic plate behaviour in the case of dynamic instability. It was assumed that the plate band under consideration is homogeneous and orthotropic. The uniperiodic structure of the plate band was related to the periodically spaced system of ribbs, the axes of which are normal to the edges. The main result of this paper is that the equations of motion for the uniperiodic plate band reduce to a certain single equation (3.4) for the plate band subjected to a time-dependent axial force provided that the plate is homogeneous and orthotropic.

Equation of motion (3.4) can be applied to the analysis of free vibrations, forced vibrations and dynamic stability (parametric resonance) of the plate band with arbitrary boundary conditions.

The frequency equation for a simply supported plate band (4.6), derived from (3.4), makes it possible to determine the static critical force and the fundamental (lower) free vibration frequency. The obtained results are a certain generalization of known solutions, cf. Bolotin (1956), and take into account the influence of the period length l on the overall plate behaviour.

On the basis of equation (4.6) a higher free vibration frequency, explicity depending on the period l, has been obtained as well. This frequency is called the structural one, and cannot be derived from the asymptotic model.

For the axial force given by $N(t) = N_0 + N_1 \cos(pt)$ we obtained equation (5.2), which can be treated as a certain generalization of the known Mathieu equation. It is a fourth-order ordinary differential equation, which reduces to the known Mathieu equation provided that the period length l is neglected.

By applying a procedure similar to that used for the investigation of the Mathieu equation, cf. Bolotin (1956), two fundamental regions of dynamic instability have been determined. The obtained results are a certain generalization of those derived from the known solution, i.e. in the proposed approach we have dealt with a certain new parameter, namely the structuralfree vibration frequency. In the determination of this frequency the definition of the modulation factor was taken into account and an extra condition on the axial force frequency was imposed on this frequency.

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Stateczność dynamiczna uniperiodycznego pasma płytowego średniej grubości

Streszczenie

Celem pracy jest zastosowanie równań uzyskanych przez Barona (2002) do analizy stateczności dynamicznej periodycznie użebrowanego, swobodnie podpartego pasma płytowego. Wyprowadzono ogólne równanie ruchu takiego pasma płytowego obciążonego zależną od czasu siłą osiową. Otrzymane wyniki zastosowano do analizy zagadnień dynamiki przy dowolnych warunkach brzegowych. Wyprowadzono równanie częstości dla pasma płytowego swobodnie podpartego. Stanowi ono pewne uogólnienie znanego równania Mathieu. Stosując tryb postępowania, jak przy rozwiązywaniu równania Mathieu, wyznaczono dwa podstawowe obszary niestateczności dynamicznej. Uzyskane wyniki są zgodne z rozwiązaniami znanymi, uwzględniają jednak zależność rozważanego problemu od wymiaru powtarzalnego segmentu płyty. W rozważaniach pojawia się dodatkowa wysoka częstość drgań własnych, której nie obejmują wyniki uzyskane przez Bolotina (1956).

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