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EXACT STEADY-STATE PROBABILITY DENSITY FUNCTIONS FOR SECOND ORDER NONLINEAR SYSTEMS UNDER EXTERNAL STATIONARY EXCITATIONS

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The main aim of this paper is to present mathematically rigorous foundations of a probabilistic analysis of vibrations in nonlinear dynamic systems driven by some stationary processes. A class of solvable Fokker-Planck equations is given and a new method is presented to obtain a probability density function of the response of a nonlinear oscillator to stationary excitations. A one-dimensional vibrating system with a nonlinear elastic force is considered.
We analyse the case when the excitation force is a stationary 2nd order stochastic process with a mean value equal to zero and a spectral density of the form

$$S_z(\omega) = \frac{S_0}{1+\omega^2\tau^2} \qquad \omega \in \mathcal{R}^1$$

where $S_0 > 0$ and τ are certain constants.
Thus, utilizing the Fokker-Planck equations, determination of the density of the three-dimensional Markov vector of the following components: displacement, velocity and acceleration of a nonlinear oscillator can be circumvented. It is presented that the density function has the following form $w^{(3)}(x, \dot{x}, ddotx) = \Phi(x, \ddot{x}) \exp[\Psi(x, \dot{x})]$ where $\Phi(\cdot,\cdot)$ and $\Psi(\cdot,\cdot)$ are analytically determined functions.

1. Introduction

Over a period of recent years the response of nonlinear oscillators to stochastic excitations has been extensively studied. In general, no exact solutions can be found. In some cases, when excitations can be idealized as Gaussian white noise, the exact solutions are obtained. The response of such a system is represented by a Markovian vector and the probability density function of the response is described by the Fokker-Planck equation (FPE). It has proved to be a quite useful tool for deriving exact solutions of problems of dynamical systems driven by white noise (see References, except Skalmierski and Tylikowski, 1972). Some results were obtained by this method to obtain exact and approximate characteristics of dynamical systems caused by broad-band random excitations (cf Caughey, 1962; Dimentberg, 1962, 1966 and 1989; Piszczek, 1970, 1971 and 1982; Piszczek and Nizioł, 1986; To and Li, 1991; Skrzypczyk, 1993).

The main aim of this paper is to give the mathematically rigorous foundations of a probabilistic analysis of vibrations in nonlinear dynamic systems driven by narrow-band stationary processes. Thus, utilizing the Fokker-Planck equations, determination of the three-dimensional Markov vector of the following components: displacement, velocity and acceleration of a nonlinear oscillator can be circumvented. Furthermore, the application of presented methods eventually leads to a relatively more general results in a finite dimensional case.

2. Preliminaries

We consider a one-dimensional vibrating system whose motion is described by a differential equation in the normalized form

$$\ddot{x}(t) + \beta\dot{x}(t) + F(x(t)) = z(t) \qquad t \in \mathcal{R}^1 \tag{2.1}$$

where $(\dot{}) = d/dt$, $\beta = \text{const} > 0$ denotes a coefficient of linear viscous damping, and the function $F(x)$, $x \in \mathcal{R}^1$ represents the characteristic of nonlinear elastic force. It is further assumed that $F(\cdot)$ is differentiable at intervals.

We consider the case when the excitation force is a stationary 2nd order stochastic process with a mean value equal to zero and a spectral density of the form

$$S_z(\omega) = \frac{S_0}{1+\omega^2\tau^2} \qquad \forall\omega \in \mathcal{R}^1 \tag{2.2}$$

where $S_0 > 0$ and τ are certain constants.

The spectral density $S_z(\cdot)$ is given by a Fourier transform

$$S_z(\omega) = \int\limits_{-\infty}^{\infty} K_z(\tau)\exp(-\mathrm{i}\omega\tau)\,d\tau \qquad \forall\omega \in \mathcal{R}^1$$

where $K_z(\cdot)$ denotes the correlation function of a certain second-order stochastic process $z(\cdot)$, stationary in a wide sense. In further analysis a stochastic process named Gaussian white noise appears, i.e. a generalized stochastic function which is a distributional derivative of the Wiener process. Its generalized correlation function will be noted as

$$K_\psi(\tau) = I\delta(\tau) \qquad \forall \tau \in \mathcal{R}^1$$

where I denotes a constant white noise intensity.

The process $z(t,\omega)$ is usually assumed to be the output of some dynamic system called filter. In our case, i.e. for spectral density given by Eq (2.2), the filter equation takes the form

$$\tau \dot{z}(t,\omega) + z(t,\omega) = \sqrt{S_0}\, \psi(t,\omega) \tag{2.3}$$

where $\psi(t,\omega)$ is a Gaussian white noise with a unit intensity (cf Skalmierski and Tylikowski, 1972).

3. Analytical solution of the Fokker-Planck equations

It is a well known fact that it is not possible to write directly the FPE for Eq (2.1). To obtain a differential equation from Eqs (2.1) and (2.3) differentiate Eq (2.1). Since the stochastic process $z(\cdot)$ is naturally not sufficiently smooth, this operation is possible in a generalized function sense only. It is assumed that the reader is familiar with the generalized calculus techniques. We obtain

$$\dddot{x} + \beta\ddot{x} + \frac{\partial F}{\partial x}\dot{x} + \frac{1}{\tau}\ddot{x} + \frac{\beta}{\tau}\dot{x} + \frac{1}{\tau}F(x) = \frac{\sqrt{S_0}}{\tau}\psi \tag{3.1}$$

It is not difficult to verify that a standard substitution for $y_1 = x, y_2 = \dot{x}$, $y_3 = z$ leads to the FPE in the form, for which it is not possible to separate variables. Following this fact we introduce more suitable variables as follows: $y_1 = x$, $y_2 = \dot{x}$, $y_3 = \ddot{x}$.

Now we can write Eq (3.1) as the system of 1st order equations

$$\begin{aligned} &\dot{y}_1 = y_2 \qquad\qquad \dot{y}_2 = y_3 \\ &\dot{y}_3 = -\beta y_3 - \frac{\partial F}{\partial y_1}y_2 - \frac{1}{\tau}y_3 - \frac{\beta}{\tau}y_2 - \frac{1}{\tau}F(y_1) + \frac{\sqrt{S_0}}{\tau}\psi \end{aligned} \tag{3.2}$$

On the basis of standard considerations (cf Piszczek, 1970, 1971 and 1982; Piszczek and Niziol, 1986; Risken, 1989) FPE corresponding to the system of differential equations (3.2) can be expressed as follows

$$\begin{aligned} y_2\frac{\partial w}{\partial y_1} + y_3\frac{\partial w}{\partial y_2} &= \frac{1+\beta\tau}{\tau}\frac{\partial}{\partial y_3}(wy_3) + y_2\Big(\frac{\partial F}{\partial y_1} + \frac{\beta}{\tau}\Big)\frac{\partial w}{\partial y_3} + \\ &+ \frac{1}{\tau}F(y_1)\frac{\partial w}{\partial y_3} + \frac{S_0}{2\tau^2}\frac{\partial^2 w}{\partial y_3^2} \end{aligned} \tag{3.3}$$

where $w = w(y_1, y_2, y_3)$, $y_1, y_2, y_3 \in \mathcal{R}^1$ denotes the probability density function.

Having in mind researches into similar problems we make the following assumption. Assume that the separation principle remains true in the form

$$w(y_1, y_2, y_3) = \Phi(y_1, y_3)\exp[\Psi(y_1, y_2)] \tag{3.4}$$

where $\Phi(\cdot,\cdot)$ and $\Psi(\cdot,\cdot)$ are certain unknown functions.

Substitution of the function (3.4) into FPE (3.3) leads to separation of two groups of variables and to the following equations

$$y_2\Phi\frac{\partial \Psi}{\partial y_1} + y_2\frac{\partial \Phi}{\partial y_1} + y_3\Phi\frac{\partial \Psi}{\partial y_2} = y_2\Big(\frac{dF}{dy_1} + \frac{\beta}{\tau}\Big)\frac{\partial \Phi}{\partial y_3} \tag{3.5}$$

$$\frac{1+\beta\tau}{\tau}\frac{\partial}{\partial y_3}(\Phi y_3) + \frac{1}{\tau}F(y_1)\frac{\partial \Phi}{\partial y_3} + \frac{S_0}{2\tau^2}\frac{\partial^2 \Phi}{\partial y_3^2} = 0 \tag{3.6}$$

Eq (3.6) can be integrated with respect to y_3. We obtain

$$\frac{1+\beta\tau}{\tau}\Phi y_3 + \frac{1}{\tau}F(y_1)\Phi + \frac{S_0}{2\tau^2}\frac{\partial \Phi}{\partial y_3} = N_1(y_1) \tag{3.7}$$

where $N_1(y_1)$ is a certain function. To solve Eq (3.7) we'll start with the homogeneous equation

$$\frac{1+\beta\tau}{\tau}\Phi y_3 + \frac{1}{\tau}F(y_1)\Phi + \frac{S_0}{2\tau^2}\frac{\partial \Phi}{\partial y_3} = 0 \tag{3.8}$$

The solution of Eq (3.8) is obtained in a routine way, it takes the form

$$\Phi_0 = N_2(y_1)\exp\Big(-\frac{\tau(1+\beta\tau)}{S_0}y_3^2 - \frac{2\tau F(y_1)}{S_0}y_3\Big) \tag{3.9}$$

where $N_2(\cdot)$ is another unknown function. We use below the following notation

$$p = -\frac{\tau(1+\beta\tau)}{S_0}y_3^2 - \frac{2\tau F(y_1)}{S_0}y_3 \tag{3.10}$$

We are looking for a general solution of the nonhomogeneous equation (3.7) accepting the classical hypotheses that

$$\Phi = C(y_1, y_3)\exp(p) \tag{3.11}$$

Substitution for the function Φ into Eq (3.7) with respect to Eq (3.8) gives the form

$$\frac{S_0}{2\tau^2}\frac{\partial C}{\partial y_3}\exp(p) = N_1(y_1) \tag{3.12}$$

Integrating Eq (3.12) we obtain

$$C(y_1, y_3) = \frac{2\tau^2}{S_0}N_1(y_1)\int_0^{y_3}\exp(-p)\,dy_3 + N_3(y_1) \tag{3.13}$$

Eqs (3.13) and (3.11) give us the final form of the solution of Eq (3.6)

$$\Phi = N_3(y_1)\exp(p) + N_4(y_1)\exp(p)\int_0^{y_3}\exp(-p)\,dy_3 \tag{3.14}$$

where $N_3(\cdot)$ and $N_4(\cdot)$ are certain unknown functions. One of these functions is determined by the normalization condition

$$\int_{R^3} w(y_1, y_2, y_3)\,dy_1 dy_2 dy_3 = 1 \tag{3.15}$$

the other one must be determined from the boundary conditions, so the problem arises as to which of the boundary conditions must be used. We consider problems where y_i, $i = 1,2,3$ extends to $\pm\infty$ (natural boundary conditions) and require that the integral (3.15) exists, therefore $N_4(y_1) \equiv 0$ for every y_1. Finally we have

$$\Phi(y_1, y_3) = N_3(y_1)\exp(p) \tag{3.16}$$

To calculate the unknown function $\Psi(\cdot,\cdot)$, we substitute for the function Φ given by Eq (3.16) into Eq (3.5). This gives

$$\begin{aligned} y_2 N_3\frac{\partial\Psi}{\partial y_1} + y_2\frac{\partial N_3}{\partial y_1} + y_2 N_3\Big(-\frac{2\tau}{S_0}\frac{dF}{dy_1}y_3\Big) + y_3 N_3\frac{\partial\Psi}{\partial y_2} = \\ = y_2\Big(\frac{dF}{dy_1} + \frac{\beta}{\tau}\Big)N_3\Big(-\frac{2\tau(1+\beta\tau)}{S_0}y_3 - \frac{2\tau F(y_1)}{S_0}\Big) \end{aligned} \tag{3.17}$$

Separating variables y_1 and y_3 we notice that the following equations must be satisfied

$$y_2 \frac{\partial N_3}{\partial y_1} = -\frac{2\tau F(y_1)}{S_0} y_2 \Big(\frac{dF}{dy_1} + \frac{\beta}{\tau}\Big) N_3 \tag{3.18}$$

and

$$y_2 N_3 \frac{\partial \Psi}{\partial y_1} + y_2 N_3 \Big(-\frac{2\tau}{S_0}\frac{dF}{dy_1} y_3\Big) + y_3 N_3 \frac{\partial \Psi}{\partial y_2} = y_2 \Big(\frac{dF}{dy_1} + \frac{\beta}{\tau}\Big) N_3 \Big(-\frac{2\tau(1+\beta\tau)}{S_0} y_3\Big) \tag{3.19}$$

Integrating Eq (3.18) we obtain

$$N_3(y_1) = N_4 \exp\Big[-\frac{2\tau}{S_0} \int \Big(\frac{dF}{dy_1} + \frac{\beta}{\tau}\Big) F(y_1)\, dy_1\Big] \tag{3.20}$$

It is obviously that integration of Eq (3.19) leads to the following result

$$\Psi(y_1, y_2) = -\frac{\beta}{S_0}\Big(1 + \beta\tau + \tau^2 \frac{dF}{dy_1}\Big) y_2^2 + N_5 \tag{3.21}$$

where N_5 is an unknown constant.

Combining the partial relations (3.16) and (3.21) we obtain the final result

$$\begin{aligned} w(y_1, y_2, y_3) \;&=\; w^{(3)}(y_1, y_2, y_3) = N_6 \exp\Big[-\frac{2\tau}{S_0} \int \Big(\frac{dF}{dy_1} + \frac{\beta}{\tau}\Big) F(y_1)\, dy_1 + \\ &-\; \frac{\beta}{S_0}\Big(1 + \beta\tau + \tau^2 \frac{dF}{dy_1}\Big) y_2^2 - \frac{\tau(1+\beta\tau)}{S_0} y_3^2 - \frac{2\tau F(y_1)}{S_0} y_3\Big] \end{aligned} \tag{3.22}$$

where N_6 is a constant. Substituting the solution (3.22) into Eq (3.15), we get

$$N_6 = \frac{\sqrt{\beta\tau(1+\beta\tau)}}{\pi S_0} w_N \tag{3.23}$$

where

$$w_N^{-1} = \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{1 + \beta\tau + \tau^2 \frac{dF}{dy_1}}} \exp\Big(-\frac{\beta\tau^2}{S_0(1+\beta\tau)} F^2(y_1) - \frac{2\beta}{S_0} \int\limits_0^{y_1} F(x)\, dx\Big)\, dy_1 \tag{3.24}$$

Notice, that if $xF(x) \geq 0 \;\; \forall x \in \mathcal{R}^1$ and $F(\cdot) \not\equiv 0$, then w_N is well-defined.

With relations (3.23) and (3.24) the probability density function (3.22) is completely determined. Integrating relation (3.22) with respect to y_3, we obtain the two-dimensional density function

$$w^{(2)}(y_1, y_2) = \int_{-\infty}^{\infty} w^{(3)}(y_1, y_2, y_3)\, dy_3 = w_N \sqrt{\frac{\beta}{\pi S_0}} \cdot$$

$$\cdot \exp\Big[-\frac{\beta\tau^2}{S_0(1+\beta\tau)} F^2(y_1) - \frac{2\beta}{S_0}\int_0^{y_1} F(x)\, dx - \frac{\beta}{S_0}\Big(1+\beta\tau+\tau^2\frac{dF}{dy_1}\Big) y_2^2\Big] \tag{3.25}$$

Observe, that if $\tau \to 0$, $S_z(\cdot) \to S_0$, the process $z(\cdot,\cdot)$ "becomes" the white noise process, and

$$w_0^{(2)}(y_1, y_2) = w_{N0}\sqrt{\frac{\beta}{\pi S_0}} \exp\Big(-\frac{2\beta}{S_0}\int_0^{y_1} F(x)\, dx - \frac{\beta}{S_0} y_2^2\Big) \tag{3.26}$$

where

$$w_{N0}^{-1} = \int_{-\infty}^{\infty} \exp\Big(-\frac{2\beta}{S_0}\int_0^{y_1} F(x)\, dx\Big)\, dy_1 \tag{3.27}$$

Similarly

$$w^{(1)}(y_1) = \int_{-\infty}^{\infty} w^{(2)}(y_1, y_2)\, dy_2 =$$

$$= w_N \frac{1}{\sqrt{1+\beta\tau+\tau^2\frac{dF}{dy_1}}} \exp\Big(-\frac{\beta\tau^2}{S_0(1+\beta\tau)} F^2(y_1) - \frac{2\beta}{S_0}\int_0^{y_1} F(x)\, dx\Big) \tag{3.28}$$

if $\tau \to 0$, we get

$$w_0^{(1)}(y_1) = w_{N0} \exp\Big(-\frac{2\beta}{S_0}\int_0^{y_1} F(x)\, dx\Big) \tag{3.29}$$

Relations (3.26), (3.27) and (3.29) are well known solutions of the FPE for nonlinear oscillator driven by white noise (cf Caughey, 1962; Piszczek, 1970, 1971 and 1982; Sobczyk, 1973; Gutowski and Świetlicki, 1986; Piszczek and Niziol, 1986).

4. Analytical solution for nonlinear elastic characteristic which is piecewise linear

Assume that a nonlinear elastic force characteristic is only linear at intervals. In details

$$F(x) = F_i(x) = \omega_i x + m_i \tag{4.1}$$

for $x_i \leq x < x_{i+1}$, $i = 1,2,...,n$, $-\infty = x_1 < x_2 < ... < x_n < x_{n+1} = +\infty$, where ω_i and m_i are constants. The motion of the vibrating system will be described by Eq (2.1).

$$w_i^{(3)}(y_1, y_2, y_3) = w_i(y_1, y_2, y_3) \quad \text{for} \quad x_i \leq y_1 < x_{i+1} \qquad y_2, y_3 \in \mathcal{R}^1 \tag{4.2}$$

where $w_i(y_1, y_2, y_3)$, $i = 1,2,...,n$ represents the probability density function determined in the ith interval of the variable y_1.

According to the results of section 2 and after necessary simple integrations we employ the following notation

$$w_i^{(3)}(y_1, y_2, y_3) = C_i \exp[-(a_i y_1^2 + b_i y_2^2 + c_i y_3^2 + d_i y_1 y_3 + e_i y_1 + g_i y_3)] \tag{4.3}$$

where due to the preceding results

$$\begin{aligned}
a_i &= \frac{\omega_i^2}{S_0}(\omega_i^2\tau + \beta) & b_i &= \frac{\beta}{S_0}(1 + \omega_i^2\tau^2 + \beta\tau) \\
c_i &= \frac{\tau}{S_0}(1 + \beta\tau) & d_i &= \frac{2\omega_i^2\tau}{S_0} \\
e_i &= \frac{2m_i}{S_0}(\omega_i^2\tau + \beta) & g_i &= \frac{2m_i\tau}{S_0}
\end{aligned} \tag{4.4}$$

for $i = 1,2,...,n$. The above results are similar to those given by Piszczek and Niziol (1986) but are obtained with less restrictive assumptions i.e. the case $\omega_i = 0$ is possible.

Integration of Eq (4.3) with respect to y_3 gives the two-dimensional density function

$$w_i^{(2)}(y_1, y_2) = \int_{-\infty}^{\infty} w_i^{(3)}(y_1, y_2, y_3)\, dy_3$$

where the result can be obtained in an analytical way

$$w_i^{(2)}(y_1, y_2) = C_i D_i \exp[-(\gamma_i y_1^2 + \delta_i y_1 + b_i y_2^2)] \tag{4.5}$$

$$\gamma_i = \frac{4a_ic_i - d_i^2}{4c_i} \qquad \delta_i = \frac{4e_ic_i - 2d_ig_i}{4c_i} \qquad D_i = \sqrt{\frac{\pi}{c_i}} \exp\left(\frac{g_i}{4c_i}\right) \tag{4.6}$$

Further integration of Eq (4.5) with respect to y_2 gives the one-dimensional density function

$$w_i^{(1)}(y_1) = \int_{-\infty}^{\infty} w_i^{(2)}(y_1, y_2)\, dy_2$$

where the result can be obtained in analytical way too

$$w_i^{(1)}(y_1) = C_iE_i \exp[-(\gamma_i y_1^2 + \delta_i y_1)] \tag{4.7}$$

$$E_i = \sqrt{\frac{\pi}{b_ic_i}} \exp\left(\frac{g_i^2}{4c_i}\right) \tag{4.8}$$

Since the elastic force characteristic is continuous, the resulting density distribution function $w(\cdot,\cdot,\cdot)$ must be continuous in all variables, too. This continuity principle gives the necessary conditions which must be satisfied

$$w_i^{(1)}(x_{i+1}) = w_{i+1}^{(1)}(x_{i+1}) \qquad i = 1, 2, ..., n-1 \tag{4.9}$$

These equations can be replaced by

$$C_iE_i\widetilde{w}_i^{(1)}(x_{i+1}) = C_{i+1}E_{i+1}\widetilde{w}_{i+1}^{(1)}(x_{i+1}) \qquad i = 1, 2, ..., n-1 \tag{4.10}$$

where

$$\widetilde{w}_i^{(1)}(y_1) = \exp[-(\gamma_i y_1^2 + \delta_i y_1)]$$

Following Eqs (4.10) we get

$$\frac{C_{i+1}}{C_i} = \frac{\widetilde{w}_i^{(1)}(x_{i+1})}{\widetilde{E}_{i+1} w_{i+1}^{(1)}(x_{i+1})} = \alpha_i \qquad i = 1, 2, ..., n-1 \tag{4.11}$$

Using these $n-1$ conditions (4.11) together with the normalization condition

$$\sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} w_i^{(1)}(y_1)\, dy_1 = 1 \tag{4.12}$$

we have the system of n linear algebraic equations with n unknown constant coefficients C_i. To complete the analytic considerations notice that

— for $\gamma_i > 0$

$$G_i = \int\limits_{x_i}^{x_{i+1}} \widetilde{w}_i^{(1)}(y_1)\,dy_1 =$$

$$= \sqrt{\frac{\pi}{\gamma_i}} \exp\Big(\frac{\delta_i}{4\gamma_i}\Big)\Big\{\operatorname{erf}\Big[\sqrt{2\gamma_i}\Big(x_{i+1}+\frac{\delta_i}{2\gamma_i}\Big)\Big] - \operatorname{erf}\Big[\sqrt{2\gamma_i}\Big(x_{i+1}+\frac{\delta_i}{2\gamma_i}\Big)\Big]\Big\}$$

— for $\gamma_i = 0$

$$G_i = -\frac{1}{\delta_i}[\exp(-\delta_i x_{i+1}) - \exp(-\delta_i x_i)]$$

where $i = 1, 2, ..., n$, and

$$\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} \exp\Big(-\frac{t^2}{2}\Big)\,dt$$

Using the introduced notation the normalization equation (4.12) takes the form

$$\sum_{i=1}^{n} C_i E_i G_i = 1 \tag{4.13}$$

From Eqs (4.11) and (4.13) we can obtain the following recursive sequence

$$\begin{aligned} C_2 &= \alpha_1 C_1 \\ C_k &= \alpha_{k-1} C_{k-1} = \alpha_1\alpha_2\cdots\alpha_{k-1}C_1 \qquad\qquad k = 3, 4, ..., n \end{aligned}$$

The simple calculations based on Eqs (4.13) and (4.14) gives the solution

$$C_1 = \frac{1}{E_1G_1 + \alpha_1E_2G_2 + \alpha_1\alpha_2E_3G_3 + ... + \alpha_1\alpha_2\cdots\alpha_{n-1}E_nG_n}$$

and application of the recursion formulae (4.14) gives the remaing coefficients C_i for $i = 2, 3, ..., n$.

4.1. Example 1

For illustration, the above theory is applied to the system, which is the nonlinear oscillator with a nonlinear spring characteristic. It is assumed that the nonlinear characteristic takes the following form

$$F(x) = \begin{cases} 0.5(x-1) & \text{for} \quad x \le -1 \\ x & \text{for} \quad -1 < x < 1 \\ 0.5(x+1) & \text{for} \quad x \ge 1 \end{cases}$$

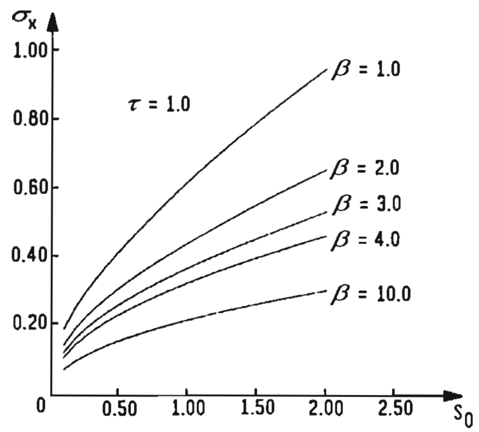

Fig. 1. Standard deviation of the response of the nonlinear oscillator with 1st type nonlinearity

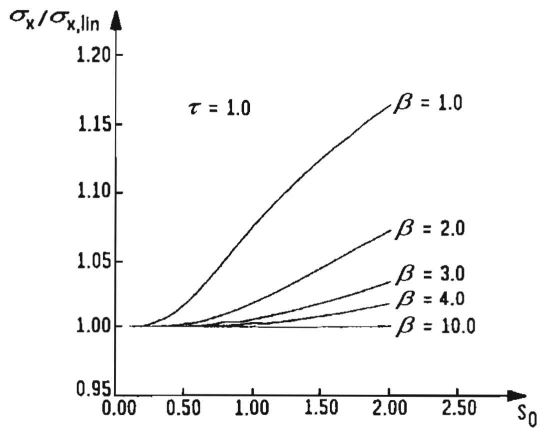

Fig. 2. Comparison of exact standard deviations of responses of the nonlinar oscillator with 1st type nonlinearity and of the linear one

and again $z(\cdot)$ is a stationary 2nd order excitation with a mean value equal to zero and the spectral density given by Eq (2.2). The resulting density functions are easy obtainable following the previous general considerations with Eqs (3.22) and (3.25), eventually particular results (4.3) $\div$ (4.8).

In Fig.1 the standard deviation σ_x of the response $x(t, \cdot)$ of the considered oscillator is presented as the function of spectral intensity S_0. The nondimensional parameters selected are: $\tau = 1.0, \beta = 1.0,\ 2.0,\ 3.0,\ 4.0,\ 10.0$.

Denote by $\sigma_{x,lin}$ the standard deviation of the response of the corresponding linear oscillator with $F(x) = x$ and the same parameter values τ and β.

The relation between standard deviations of responses of the nonlinear oscillator and the linear one are depicted in Fig.2.

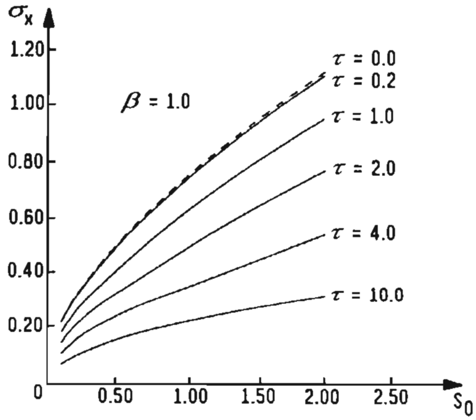

Fig. 3. Standard deviation of the response of the nonlinear oscillator with 1st type nonlinearity, - - - white noise

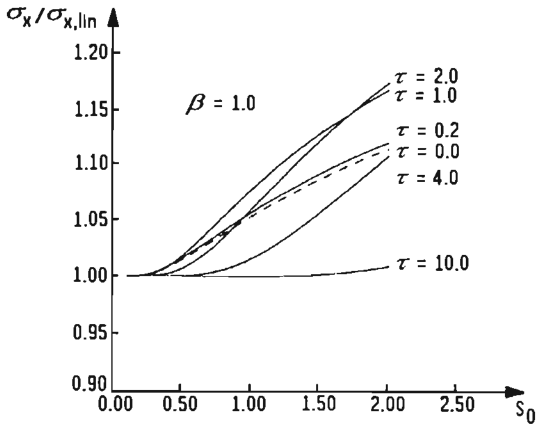

Fig. 4. Comparison of exact standard deviations of responses of the nonlinar oscillator with 1st type nonlinearity and of the linear one, - - - white noise

In Fig.3 the standard deviation σ_x of the response $x(t,\cdot)$ of the considered oscillator is presented as the function of spectral intensity S_0 for a different set of selected parameters: $\beta = 1.0, \tau = 0.0, 0.2, 1.0, 2.0, 4.0, 10.0$. Notice that by a dashed line is denoted the case of white noise excitation, and compare it with responses for other $\tau \neq 0$. Since the white noise approximation is widely used in various technical applications, it can be stated from the presented figure, what is the range of admissible τ values.

In Fig.4 the similar analysis is presented for relative values of standard deviations of a nonlinear system with respect to a linear one. Notice the strange character of changes of the relative value $\sigma_x/\sigma_{x,lin}$ with respect to $\tau \in [0.0, 10.0]$. It is a result of the resonant character of a system under considerations.

4.2. Example 2

To show the applicability of the method we discuss further the nonlinear Duffing oscillator with the spring characteristic in the form

$$F(x) = k_1 x + k_2 x^3$$

where k_1, k_2 are certain constants. Resulting probability characteristics are well known in the case of white noise excitation (cf Caughey, 1962; Piszczek, 1970, 1971 and 1982; Sobczyk, 1973; Gutowski and Świetlicki, 1986; Piszczek and Nizioł, 1986; To and Li, 1991). In the case of other stationary exciting processes analytical results are not known, or obtained for "small" values of τ only (cf Dimentberg, 1962; Piszczek and Nizioł, 1986).

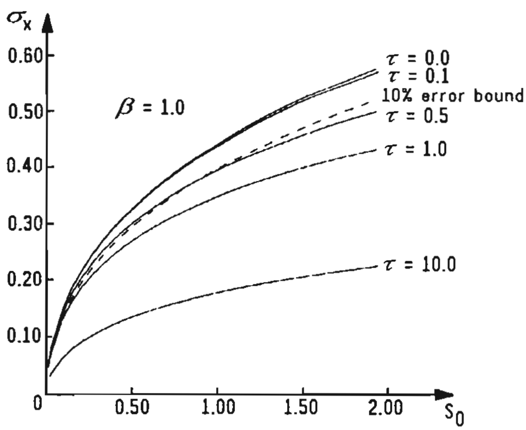

Fig. 5. Standard deviation of the response of the Duffing oscillator with parameters $k_1 = k_2 = 1.0$, $\tau = 0.0$ – white noise, – – – error bound in white noise approximation

The resulting density functions are easy obtainable following Eqs (3.22) and (3.25), eventually Eqs (4.3) ÷ (4.8). Focus our attention on a standard deviation of a response of the considered nonlinear system. In Fig.5 the standard deviation of the dimensionless response of the nonlinear oscillator is plotted, for different values of parameters: $\beta = 1.0$, $\tau = 0.0$ (white noise excitation), 0.1, 0.5, 1.0, 10.0. By a dashed line the 10% error bound of "small τ" approximation is determined to enhance an error of the previous results (cf Dimentberg, 1962; Piszczek and Niziol, 1986).

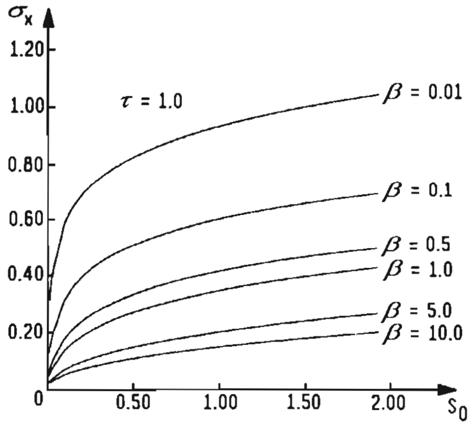

Fig. 6. Standard deviation of the response of the Duffing oscillator with parameters $k_1 = k_2 = 1.0$

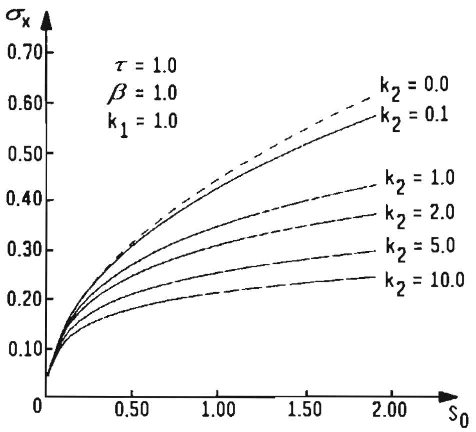

Fig. 7. Standard deviation of the response of the Duffing oscillator,– – – linear case

In Fig.6 the exact standard deviations of the nonlinear oscillator are presented varying the value of damping ratios $\beta = 0.1$, 0.5, 1.0, 5.0, 10.0 and for $\tau = 1.0$.

In Fig.7 and Fig.8 the exact standard deviations of the nonlinear oscillator are depicted as the functions of spectral intensity S_0, varying the values of $k_2 = 0.0$ (linear system), 0.1, 1.0, 2.0, 5.0, 10.0, $k_1 = 1.0$ and $k_1 = 0.0$, 0.1, 1.0, 2.0, 5.0, 10.0, $k_2 = 1.0$ respectively, for $\beta = \tau = 1.0$.

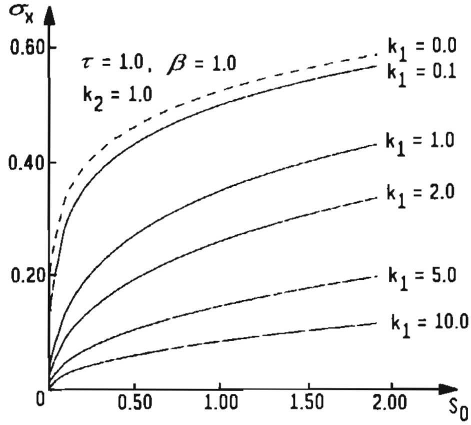

Fig. 8. Standard deviation of the response of the Duffing oscillator

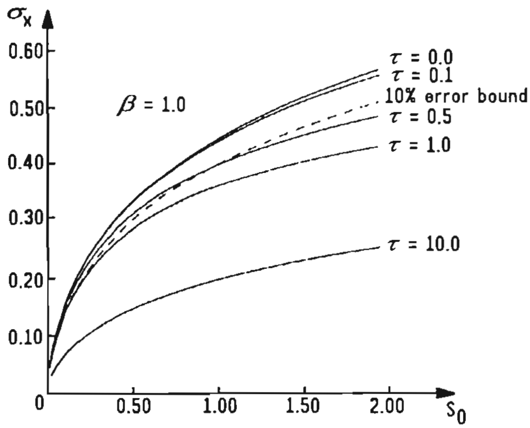

Fig. 9. Standard deviation of the response of the quintic oscillator with parameters $k_1 = k_2 = 1.0$, $\tau = 0.0$ – white noise, – – – error bound in white noise approximation

4.3. Example 3

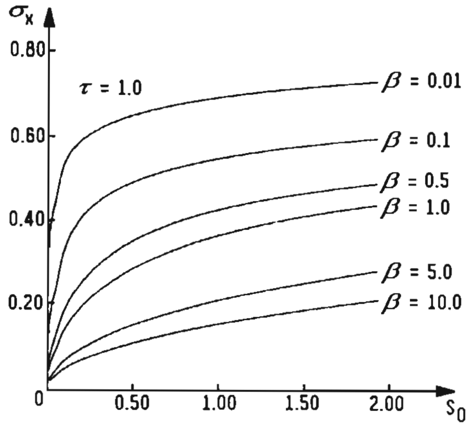

Fig. 10. Standard deviation of the response of the quintic oscillator with parameters $k_1 = k_2 = 1.04$

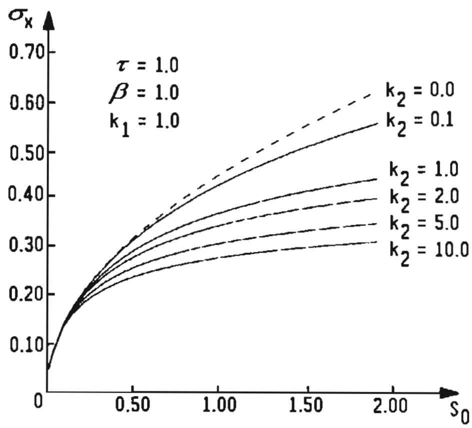

Fig. 11. Standard deviation of the response of the quintic oscillator, – – – linear case

Consider another oscillator, governed by the spring characteristic equation

$$F(x) = k_1 x + k_2 x^5$$

i.e. the quintic oscillator.Exact probability characteristics are known for white noise excitation (cf To and Li, 1991). Exact response characteristics of other stationary solutions are not known. The analysis similar to that decribed in details in Example 2 is illustrated in Fig.9 ÷ Fig.12.

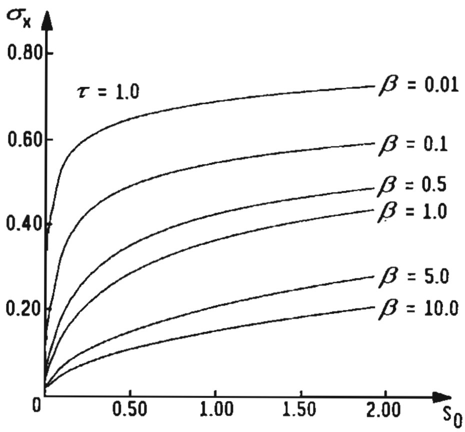

Fig. 12. Standard deviation of the response of the quintic oscillator

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Ścisłe funkcje gęstości prawdopodobieństwa w stanie ustalonym dla nieliniowych układów drugiego rzędu poddanych zewnętrznym wymuszeniom stacjonarnym

Streszczenie

Podstawowym celem pracy jest przedstawienie matematycznie ścislej analizy probabilistycznej drgań nieliniowego ukladu dynamicznego poddanego wymuszeniu, którym jest pewien proces stochastyczny stacjonarny, 2-go rzędu. Analizowana jest taka klasa równań Fokkera-Plancka, dla których możliwe jest podanie rozwiązania analitycznego. Podana jest częściowo nowa metoda analizy równań Fokkera-Plancka, która pozwala na uzyskanie funkcji gęstości prawdopodobieństwa stacjonarnej odpowiedzi nieliniowego oscylatora poddanego wymuszeniu stacjonarnemu. Analizowany jest jednowymiarowy uklad drgający z nieliniową częścią sprężystą w przypadku gdy siła wymuszająca jest stacjonarnym w szerokim sensie procesem stochastycznym z wartością oczekiwaną równą zeru i gęstością spektralną postaci

$$S_z(\omega) = \frac{S_0}{1+\omega^2\tau^2} \qquad \omega \in \mathcal{R}^1$$

gdzie $S_0 > 0$ i τ są pewnymi stalymi.

Pokazano, że wykorzystując równanie Fokkera-Plancka możliwe jest określenie gęstości prawdopodobieństwa trójwymiarowego wektora Markowa ze skladowymi: przemieszczeniem, prędkością i przyspieszeniem nieliniowego oscylatora. Wykazano dalej, że ta funkcja gęstości prawdopodobieństwa ma następującą postać $w^{(3)}(x,\dot{x},ddotx) = \Phi(x,\ddot{x})\ \exp[\Psi(x,\dot{x})]$ gdzie $\Phi(\cdot,\cdot)$ i $\Psi(\cdot,\cdot)$ są funkcjami, które można określić analitycznie.

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