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ON THE WAVE PROPAGATION IN MICRO-INHOMOGENEOUS MEDIA[1]

Sylwester Konieczny

Małgorzata Woźniak

Faculty of Civil Engineering, Architecture and Environmental Engineering

Technical University of Łódź

The wave propagation problem in periodic fibre-reinforced linear-elastic materials is investigated. The approach is based on the equations of refined macrodynamics proposed by Woźniak (1993) and generalizes the results obtained previously by Mielczarek and Woźniak (1995). Its main feature is a possibility of obtaining resulting formulas for the spectral lines and the phase velocities in a simple analytical form.

1. Preliminaries

The problems of wave propagation in periodic composites have been studied in a series of papers by Achenbach and Sun (1972), Hjalmars and Fischer-Hjalmars (1981), Abudi (1981), Tolf (1982), Żórawski (1982) and others. In most cases the spectral lines as well as the phase and group velocities were obtained by numerical solutions to boundary value problems related to the representative array of the medium. In this paper an alternative approach to this problem, based on the equations of the refined macrodynamics, proposed by Woźniak (1993), is investigated. The considerations will be restricted to the waves propagating in the plane $0x_1x_2$, normal to fibres. The fibres are assumed to be parallel, periodically distributed and have circular cross-sections; the representive array A of the composite on $0x_1x_2$-plane is shown in Fig.1. Both the matrix and the reinforcement are made of homogeneous, isotropic, linear-elastic materials and the perfect bonding between constituents is taken

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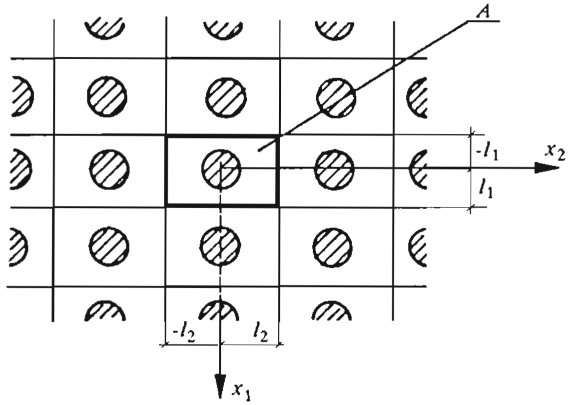

Fig. 1. Cross-section of the fibrous composite and its representative array A

into account. The analysis will be carried out within the framework of the macro-wave approximation in which higher powers of the dimensionless wave number $q = kl$ ($k = 2\pi/L$ being the wave number and $l \equiv 2\sqrt{(l_1)^2 + (l_2)^2}$ is the maximum characteristic length dimension of the representative array) are small compared to 1 and can be neglected. The main attention will be devoted to the waves propagating along x_1-axis. On these assumptions the results found are given in a simple form of explicit interrelations between the frequency $\tilde{\omega}$ and the wave number k, representing three basic modes of the spectral lines both for the longitudinal and transversal waves. Hence the obtained dispersion relations constitute a certain generalization of formulae derived previously for transversal waves by Mielczarek and Woźniak (1995).

2. Foundations

In order to specify the general form of governing equations of the refined macroelastodynamics (cf Woźniak (1993)) for the problem under consideration we shall introduce two micro-oscillatory shape functions $h_1 = h_1(x_1, x_2)$, $h_2 = h_2(x_1, x_2)$ which are A-periodic, $A = (-l_1, l_1) \times (-l_2, l_2)$, continuous, and describing, from the qualitative viewpoint, the expected disturbances of the displacements due to micro-heterogeneity of the medium. The diagrams of function $h_1(x_1, x_2)$ on the matrix-fiber interface and for $x_2 = a$, $a \in (0, r)$, are given in Fig.2; at the same time $h_1(x_1, z) \equiv 0$ for every $z \in [r, l_1]$ and $h_1(x_1, x_2) = h_1(x_1, -x_2)$ for every $(x_1, x_2) \in A$. The extremal values of

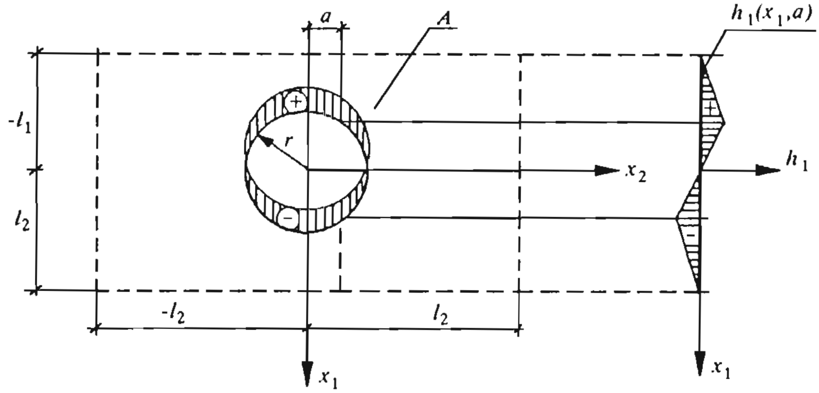

Fig. 2. The diagrams of function $h_1(x_1, x)$ for $x_2 = a$, $a \in [0, r]$ and on the matrix-fiber interface

$h_1(x_1, x_2)$ are for $(x_1, x_2) = (\pm r, 0)$, being equal to l_1. The form of function $h_2(x_2, x_1)$ can be obtained from $h_1(x_1, x_2)$ by interchanging subscripts 1,2 in exact formulas defining $h_1(x_1, x_2)$ (hence l_1, l_2 have to be replaced by l_2, l_1, respectively, etc.). The micro-macro localization hypothesis of the refined macrodynamics yields the following approximation for the displacement fields $u_i(\boldsymbol{x}, \tau)$, $i = 1,2$, $\boldsymbol{x} \equiv (x_1, x_2)$ at the instant τ (cf Eq (3.1) in Woźniak (1993))

$$u_i(\boldsymbol{x}, \tau) = U_i(\boldsymbol{x}, \tau) + h_1(\boldsymbol{x})Q_i^1(\boldsymbol{x}, \tau) + h_2(\boldsymbol{x})Q_i^2(\boldsymbol{x}, \tau) \qquad i = 1,2 \tag{2.1}$$

where macrodisplacements U_i and macrocorrectors Q_i^1, Q_i^2 are dynamic variables. At the same time these variables are macro functions i.e. they satisfy conditions

$$\begin{gathered}\forall \boldsymbol{x}, \boldsymbol{z}: \ \boldsymbol{x} - \boldsymbol{z} \in A \ \Rightarrow \ |F(\boldsymbol{x}) - F(\boldsymbol{z})| < \varepsilon_F \\ F \in \{U_i, Q_i^1, Q_i^2, U_{i,j}, Q_{i,j}^1, Q_{i,j}^2, \dot{U}_i, ...\}\end{gathered} \tag{2.2}$$

where values $\mathcal{O}(\varepsilon_F)$ in the course of macro-modelling are rejected as negligibly small. For more detailed information the reader is referred to the above references. Let us denote by ρ and λ, μ the mass density and the Lamé modulae, attaining for fibres and a matrix constant but different values ρ_F, λ_F, μ_F and ρ_M, λ_M, μ_M, respectlively. Setting $\kappa \equiv \lambda + 2\mu$ and introducing the averaging operator

$$< f >= \frac{1}{4l_1 l_2} \int\limits_A f(x_1, x_2) \, dx_1 dx_2$$

for an arbitrary A-periodic integrable function f, from Eqs (5.1) in Woźniak (1993), we obtain (summation over $b = 1,2$ holds)

$$
\begin{aligned}
&<\rho> \ddot{U}_1 - <\kappa> U_{1,11} - <\mu> U_{1,22} - <\lambda+\mu> U_{2,12} - <\kappa h_{1,1}> Q^1_{1,1} + \\
&- <\lambda h_{2,2}> Q^2_{2,1} - <\mu h_{2,2}> Q^2_{1,2} - <\mu h_{1,1}> Q^1_{2,2} = 0 \\[6pt]
&<\rho> \ddot{U}_2 - <\kappa> U_{2,22} - <\mu> U_{2,11} - <\lambda+\mu> U_{1,12} - <\kappa h_{2,2}> Q^2_{2,2} + \\
&- <\lambda h_{1,1}> Q^1_{1,2} - <\mu h_{1,1}> Q^1_{2,1} - <\mu h_{2,2}> Q^2_{1,1} = 0 \\[6pt]
&<\rho h_a h_b> \ddot{Q}^b_1 + <\kappa h_{a,1} h_{b,1} + \mu h_{a,2} h_{b,2}> Q^b_1 + <\lambda h_{a,1} h_{b,2} + \mu h_{a,2} h_{b,1}> Q^b_2 + \\
&+ <\kappa h_{a,1}> U_{1,1} + <\lambda h_{a,1}> U_{2,2} + <\mu h_{a,2}> (U_{1,2} + U_{2,1}) = 0 \\[6pt]
&<\rho h_a h_b> \ddot{Q}^b_2 + <\kappa h_{a,2} h_{b,2} + \mu h_{a,1} h_{b,1}> Q^b_2 + <\lambda h_{a,2} h_{b,1} + \mu h_{a,1} h_{b,2}> Q^b_1 + \\
&+ <\kappa h_{a,2}> U_{2,2} + <\lambda h_{a,2}> U_{1,1} + <\mu h_{a,1}> (U_{2,1} + U_{1,2}) = 0 \qquad a = 1,2
\end{aligned}
\tag{2.3}
$$

where $<\rho h_a h_b>= 0$ if $a \neq b$. Eqs (2.3) represent the system of governing equations of the refined macrodynamics for macrodisplacements U_i and macrocorrectors Q^a_i; for the sake of simplicity body forces have been neglected. The above equations constitute foundations of the subsequent analysis of wave propagation. For the detailed discussion of Eqs (2.3) the reader is referred to the papers on the refined macrodynamics.

3. Analysis

In the sequel we confine ourselves to the analysis of waves propagating along x_1-axis. To this end we assume $U_i = U_i(x_1, \tau)$, $Q^a_i = Q^a_i(x_1, \tau)$. On this assumption by the direct calculations it can be shown that the system of governing equations (2.3) is divided into two independent systems for U_1, Q^1_1, Q^2_2 and for U_2, Q^1_2, Q^2_1. The first of them will be used to the investigation of longitudinal waves and the second one describes propagation of transversal waves. Moreover, from the analytical viewpoint both systems have a similar structure and can be written down in one compact form. To this end we shall introduce the following denotations for unknown functions

$$U \equiv \left\{ \begin{array}{c} U_1(x_1,\tau) \\ U_2(x_1,\tau) \end{array} \right\} \qquad Q^1 \equiv \left\{ \begin{array}{c} Q_1^1(x_1,\tau) \\ Q_2^1(x_1,\tau) \end{array} \right\}$$
$$Q^2 \equiv \left\{ \begin{array}{c} Q_2^2(x_1,\tau) \\ Q_1^2(x_1,\tau) \end{array} \right\} \tag{3.1}$$

and for averaged modulae

$$\gamma \equiv \left\{ \begin{array}{c} <\kappa> \\ <\mu> \end{array} \right\} \qquad \gamma_1 \equiv \left\{ \begin{array}{c} <\kappa h_{1,1}> \\ <\mu h_{1,1}> \end{array} \right\} \qquad \gamma_2 \equiv \left\{ \begin{array}{c} <\lambda h_{2,2}> \\ <\mu h_{2,2}> \end{array} \right\}$$
$$\gamma_{11} \equiv \left\{ \begin{array}{c} <\kappa(h_{1,1})^2 + <\mu(h_{1,2})^2> \\ <\mu(h_{1,1})^2 + <\kappa(h_{1,2})^2> \end{array} \right\} \qquad \gamma_{12} \equiv \left\{ \begin{array}{c} <\lambda h_{1,1}h_{2,2}> \\ <\mu h_{1,1}h_{2,2}> \end{array} \right\} \tag{3.2}$$
$$\gamma_{22} \equiv \left\{ \begin{array}{c} <\kappa(h_{2,2})^2 + <\mu(h_{2,1})^2> \\ <\mu(h_{2,2})^2 + <\kappa(h_{2,1})^2> \end{array} \right\} \qquad \tilde{\rho} \equiv <\rho>$$

It can be seen that the values of modulae (3.2) are independent of the length dimensions l_1, l_2, r of the representative array A, i.e., they are invariant under arbitrary rescalling microstructure down by setting $l \to 0$ and keeping $l_1/l, l_2/l$ and r/l constant, where $l \equiv 2\sqrt{(l_1)^2 + (l_2)^2}$ is the maximum characteristic length dimension of A. On the contrary, the micro-inertial modulae $<\rho h_a h_b>$, $a = b$, in Eqs (2.3) are of the order $\mathcal{O}(l^2)$. Hence, setting

$$\rho_{11} \equiv \frac{<\rho(h_1)^2>}{l^2} \qquad \rho_{22} \equiv \frac{<\rho(h_2)^2>}{l^2} \tag{3.3}$$

we introduce micro-inertial modulae ρ_{11}, ρ_{22} which behave like constant for $l \to 0$. On the assumption that macro-functions U_i, Q_i^a depend only on x_1, τ, $U_i = U_i(x_1,\tau)$, $Q_i^a = Q_i^a(x_1,\tau)$, and introducing denotations (3.1) $\div$ (3.3), Eqs (2.3) can be represented by two independent systems of equations, which can be jointly written down in the form

$$\begin{aligned} &\tilde{\rho}\ddot{U} - \gamma U_{,11} - \gamma_1 Q^1_{,1} - \gamma_2 Q^2_{,2} = 0 \\ &l^2 \rho_{11}\ddot{Q}^1 + \gamma_{11}Q^1 + \gamma_{12}Q^2 + \gamma_1 U_{,1} = 0 \\ &l^2 \rho_{22}\ddot{Q}^2 + \gamma_{12}Q^1 + \gamma_{22}Q^2 + \gamma_2 U_{,2} = 0 \end{aligned} \tag{3.4}$$

It has to be emphasized that Eqs (3.4) involve in the explicit form the microstructure length parameter l, i.e., all coefficients in Eqs (3.4) are independent of l (are invariant under rescalling $l \to 0$). At the same time we have to keep in mind that Eqs (2.3) and hence Eqs (3.4) have a physical sense only if U_i, Q_i^1, Q_i^2, $i = 1,2$ are macro functions, satisfying the condition (2.2). These two facts play a crucial role in the approach below.

The subsequent part of analysis is similar to that proposed by Mielczarek and Woźniak (1995); that is why we restrict ourselves to the main points of approach. Looking for solution to Eqs (3.4) in the form

$$U = A\cos(\check{\omega}\tau - kx_1) \qquad Q^1 = B^1\sin(\check{\omega}\tau - kx_1) \qquad Q^2 = B^2\sin(\check{\omega}\tau - kx_1) \tag{3.5}$$

where A, B^1, B^2 are arbitrary constants, we obtain non trivial solutions only if the determinant of the resulting system of linear algebraic equations for A, B^1, B^2 is equal to zero. This condition represents the dispersion relation between the wave number k and the frequency $\check{\omega}$. Under denotations

$$\begin{aligned}
&\tilde{\alpha} \equiv \frac{\gamma}{\rho} && \alpha \equiv \frac{\gamma_{11}}{\rho_{11}} + \frac{\gamma_{22}}{\rho_{22}} \\
&\beta \equiv \frac{\gamma_{11}\gamma_{22} - (\gamma_{12})^2}{\rho_{11}\rho_{22}} && \nu_1 \equiv \frac{1}{\tilde{\rho}}\Big[\frac{(\gamma_1)^2}{\rho_{11}} + \frac{(\gamma_2)^2}{\rho_{22}}\Big] \\
&\nu_2 \equiv \frac{\gamma_{11}(\gamma_2)^2 + \gamma_{22}(\gamma_1)^2 - 2\gamma_1\gamma_2\gamma_{12}}{\tilde{\rho}\rho_{11}\rho_{22}}
\end{aligned} \tag{3.6}$$

this relation reads

$$l^4\check{\omega}^6 - (\tilde{\alpha}k^2l^4 + \alpha l^2)\check{\omega}^4 + [(\tilde{\alpha}\alpha - \nu_1)k^2l^2 + \beta]\check{\omega}^2 - (\tilde{\alpha}\beta - \nu_2)k^2 = 0 \tag{3.7}$$

where l is treated as a small parameter. For the detailed discussion and a method of finding the solutions to Eq (3.7) cf Mielczarek and Woźniak (1995). Setting

$$\begin{aligned}
\gamma^{\text{eff}} &\equiv \gamma - \frac{\gamma_{11}(\gamma_2)^2 + \gamma_{22}(\gamma_1)^2 - 2\gamma_1\gamma_2\gamma_{12}}{\det \gamma_{ij}} \\
\eta &\equiv \frac{\rho_{11}}{\tilde{\rho}}\Big(\frac{\gamma_{22}\gamma_1 - \gamma_{12}\gamma_2}{\det \gamma_{ij}}\Big)^2 + \frac{\rho_{22}}{\tilde{\rho}}\Big(\frac{\gamma_{11}\gamma_2 - \gamma_{12}\gamma_1}{\det \gamma_{ij}}\Big)^2 \\
\delta^2 &\equiv \Big(\frac{\gamma_{11}}{\rho_{11}} - \frac{\gamma_{22}}{\rho_{22}}\Big)^2 + 4\frac{(\gamma_{12})^2}{\rho_{11}\rho_{22}} \\
\det \gamma_{ij} &= \gamma_{11}\gamma_{22} - (\gamma_{12})^2
\end{aligned} \tag{3.8}$$

we obtain following formulae for three modes of the spectral lines, constituting solutions to Eq (3.7)

$$
\begin{aligned}
(\breve{\omega}_1)^2 &= \frac{\gamma^{\text{eff}}}{\widetilde{\rho}}k^2\Big[1-\eta(kl)^2\Big]+\mathcal{O}(q^6) \qquad\qquad q\equiv kl=\frac{2\pi l}{L}\\
(\breve{\omega}_2)^2 &= \frac{\alpha-\delta}{2l^2}+\frac{2(\alpha-\delta)\nu_1-4\nu_2}{3(\alpha-\delta)^2-4(\alpha-\delta)\alpha+4\beta}k^2+\mathcal{O}(q^4)\\
(\breve{\omega}_3)^2 &= \frac{\alpha+\delta}{2l^2}+\frac{2(\alpha+\delta)\nu_1-4\nu_2}{3(\alpha+\delta)^2-4(\alpha+\delta)\alpha+4\beta}k^2+\mathcal{O}(q^4)
\end{aligned}
\tag{3.9}
$$

Using the procedure similar to that applied by Mielczarek and Woźniak (1995) it can be shown that

$$
\begin{aligned}
&0<\gamma^{\text{eff}}<\gamma \qquad\qquad \det\gamma_{ij}>0\\
&3(\alpha\pm\delta)^2-4(\alpha\pm\delta)\alpha+4\beta>0
\end{aligned}
$$

Bearing in mind that the solutions to Eqs (3.4), given by formulae (3.5) have to be A-macro functions, we conclude that $q=kl$ should be small compared to 1. Since terms of the order $\mathcal{O}(l^2)$ are retained in Eqs (3.4) then we introduce the so-called macro-wave approximation by assuming $1+\mathcal{O}(q^4)\cong 1$. Hence, terms $\mathcal{O}(q^6)$ in the first from Eqs (3.9), as well as terms $\mathcal{O}(q^4)$ in the second and third equation, can be neglected

$$
\begin{aligned}
(\breve{\omega}_1)^2 &= \frac{\gamma^{\text{eff}}}{\widetilde{\rho}}k^2\Big[1-\eta(kl)^2\Big]\\
(\breve{\omega}_2)^2 &= \frac{\alpha-\delta}{2l^2}+\frac{2(\alpha-\delta)\nu_1-4\nu_2}{3(\alpha-\delta)^2-4(\alpha-\delta)\alpha+4\beta}k^2\\
(\breve{\omega}_3)^2 &= \frac{\alpha+\delta}{2l^2}+\frac{2(\alpha+\delta)\nu_1-4\nu_2}{3(\alpha+\delta)^2-4(\alpha+\delta)\alpha+4\beta}k^2
\end{aligned}
\tag{3.10}
$$

Eqs (3.10) also yield the simple formulae for the phase velocities $c_i=\omega_i/k$, $i=1,2,3$. If $l_1=l_2$ then Eqs (3.10) reduce to the form

$$
\begin{aligned}
(\breve{\omega}_1)^2 &= \frac{\gamma^{\text{eff}}}{\widetilde{\rho}}k^2\Big[1-\frac{2\rho_{11}}{\widetilde{\rho}}\Big(\frac{\gamma_1}{\gamma_{11}-\gamma_{12}}\Big)^2(kl)^2\Big]\\
(\breve{\omega}_2)^2 &= \frac{\gamma_{11}-\gamma_{12}}{\rho_{11}}\frac{1}{l^2}+\frac{(\gamma_1)^2}{\widetilde{\rho}(\gamma_{11}-\gamma_{12})}k^2\\
(\breve{\omega}_3)^2 &= \frac{\gamma_{11}+\gamma_{12}}{\rho_{11}}\frac{1}{l^2}+\frac{(\gamma_1)^2}{\widetilde{\rho}(\gamma_{11}+\gamma_{12})}k^2
\end{aligned}
\tag{3.11}
$$

The formulae (3.10) for the spectral lines and their special form given in Eqs (3.11) hold for both longitudinal and transversal waves, represented by upper and lower terms, respectively, in definitions (3.2). In the paper by Mielczarek and Woźniak (1995) only transversal waves were considered; however, the discussion of the obtained results is quite similar to that given in the paper quoted above and will be not repeated here.

It can be seen that main advantage of the proposed approach lies in the simple form of the resulting formulae (3.10), (3.11). It has to be remembered that Eqs (3.10), (3.11) are valid for values of $q = kl$ satisfying macro-wave approximation which can be assumed in the form $1 + q^4 \cong 1$. Moreover, by introducing in Eq (2.1) only two micro-shape functions $h_1(x)$, $h_2(x)$ we have formulated a certain first approximation for the problem under consideration. On the other hand, the comparison of results obtained within this approximation and the exact solutions which are known for the laminated media (cf Mielczarek and Woźniak (1995)) shows that both results nearly coincide.

At the end of this contribution we present some numerical results. The calculations were carried out for $l_1 = l_2 = 4r$, $r/l_1 = r/l_2 = 0.25$, $\nu_F = 0.4$, $\nu_M = 0.3$, $E_F/E_M = \psi$, $\psi = 5$, 10, 25 and $\rho_R/\rho_M = 1.5$ by using definitions (3.2), (3.3), (3.6) and (3.8). The diagrams of spectral lines were derived from formulas (3.10), (3.11) and are shown in Fig.3 and Fig.4.

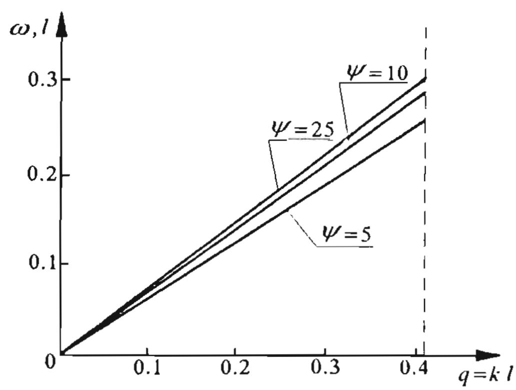

Fig. 3. The diagrams of spectral lines for $\omega = \omega_1$

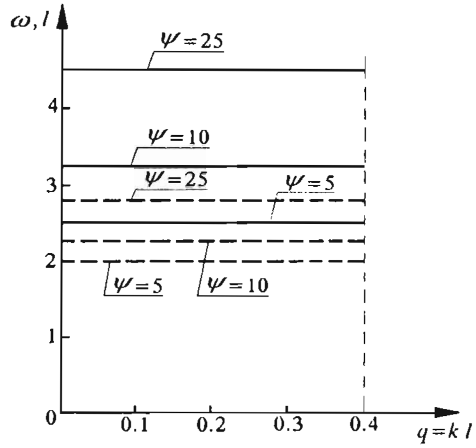

Fig. 4. The diagrams of spectral lines for $\omega = \omega_2$ (broken lines) and $\omega = \omega_3$ (continuous lines)

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O propagacji fal w ośrodkach mikro-niejednorodnych

Streszczenie

W opracowaniu uogólniono przedstawione w pracy [4] podejście do problemu propagacji fal w periodycznie zbrojonych włóknistych kompozytach. Zaproponowana metoda korzysta z równań rozszerzonej makro-dynamiki, [6,7], umożliwiając uzyskanie równań lini spektralnych i prędkości fazowych w prostej analitycznej postaci.

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