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GENERALIZED MEIR-KEELER TYPE ψ-CONTRACTIVE MAPPINGS AND APPLICATIONS TO COMMON SOLUTION OF INTEGRAL EQUATIONS

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Abstract. The goal of the present article to introduce the notion of generalized Meir-Keeler type ψ-contractions and prove some coupled common fixed point results for such type of contractions. The theorems proved herein extend, generalize and improve some results of the existing literature. Several examples and an application to integral equations are also given in order to illustrate the genuineness of our results.

1. Introduction and Preliminaries

The Meir-Keeler contraction defined in 1969 by Meir and Keeler [13] is one of the most significant generalizations of Banach contraction principle [1]. Owing to it's utility, generality and effectiveness, the result of Meir and Keeler [13] remains a novel result in metric fixed point theory. In recent years, many authors extended and generalized this result in different ways and by now there exists extensive literature on this theme. To mention a few, we recall [3, 7, 15, 16] and references cited therein.

Guo and Lakshmikantham [8] established some coupled fixed point theorems which has attracted the attention of many researchers (e.g. [5–7,9–12] and references therein). Bhaskar and Lakshmikantham [2] introduced the notion of mixed monotone mapping to prove results on coupled fixed points. As an application, they also proved the existence and uniqueness of solution for a periodic boundary value problem associated to a first order ordinary differential equation. Recently, Ding et al. [6] introduced the notion of weakly increasing mappings with two variables and established several coupled common fixed point theorems for these mappings in ordered metric spaces.

In this paper, we introduce the notion of generalized Meir-Keeler type ψ-contractions and prove some coupled common fixed point theorems for such contractions using weakly increasing property instead of mixed monotone property. Our results extend, generalize and improve several results of the existing literature. Several interesting consequences of our theorems are derived besides furnishing an example. As an application of the results presented herein, we discuss the existence of the common solution for a system of integral equations.

We start by recalling some definitions and notions. In the sequel, the letters $\mathbb{R}, \mathbb{R}^+$and $\mathbb{N}$ will denote the set of all real numbers, the set of all non negative real numbers and the set of all natural numbers, respectively.

Definition 1.1 ([2]). *An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $F : X^2 \to X$ if $x = F(x, y)$ and $y = F(y, x)$.*

Definition 1.2 ([6]). *An element $(x, y) \in X^2$ is called a coupled common fixed point of mappings $F, G : X^2 \to X$ if $F(x, y) = G(x, y) = x$ and $F(y, x) = G(y, x) = y$. We denote the set of all coupled common points of F and G by $\mathcal{F}(F, G)$.*

Definition 1.3 ([2]). *Let $(X, \preceq)$ be a partially ordered set and $F : X^2 \to X$. We say that F has the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and monotone nonincreasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

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and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow F(x, y_2) \preceq F(x, y_1).$$

Definition 1.4 ([6]). *Let $(X, \preceq)$ be a partially ordered set. Two mappings $F, G : X^2 \to X$ are said to be weakly increasing if*

$$F(x, y) \preceq G(F(x, y), F(y, x)) \quad \text{and} \quad G(x, y) \preceq F(G(x, y), G(y, x)),$$

hold for all $(x, y) \in X^2$.

Example 1.1. *Let $X = [1, +\infty)$ be endowed with the usual ordering $\leq$ and $F, G : X^2 \to X$ be given by $F(x, y) = x^2 + 2y$ and $G(x, y) = 2x + y^2$. Then, for all $(x, y) \in X^2$,*

$$\begin{aligned} F(x, y) = x^2 + 2y &\leq G(F(x, y), F(y, x)) \\ &= G(x^2 + 2y, y^2 + 2x) = 2(x^2 + 2y) + (y^2 + 2x)^2 \end{aligned}$$

and

$$\begin{aligned} G(x, y) = 2x + y^2 &\leq F(G(x, y), G(y, x)) \\ &= F(2x + y^2, 2y + x^2) = \left(2x + y^2\right)^2 + 2\left(2y + x^2\right). \end{aligned}$$

Thus, F and G are two weakly increasing mappings with respect to $\leq$.

On the lines of [4], we denote by Ψ the family of all functions $\psi : \mathbb{R}^{+^4} \to \mathbb{R}^+$ such that

(ψ_1) ψ is nondecreasing and continuous in each coordinate;
(ψ_2) $\psi(t, t, t, t) \leq t$ for all $t > 0$;
(ψ_3) $\psi(t_1, t_2, t_3, t_4) = 0$ iff $t_1 = t_2 = t_3 = t_4 = 0$.

In the rest of the paper, we denote by $(X, \preceq, d)$ an ordered metric space wherein $\preceq$ is a partial order on the set X while d is a metric on X. In addition, we say that $(x, y) \in X^2$ is comparable to $(u, v) \in X^2$ if $x \preceq u$ and $y \preceq v$ or $u \preceq x$ and $v \preceq y$. For brevity, we show this with $(x, y) \preceq (u, v)$ or $(x, y) \succeq (u, v)$.

(X^2, δ) is a metric space under the following metric:

$$\delta((x, y), (u, v)) := d(x, u) + d(y, v),$$

for all $(x, y), (u, v) \in X^2$. By the definition of δ, it is obvious that

$$\delta((x, y), (u, v)) = \delta((y, x), (v, u)).$$

Definition 1.5. *Let $(X, \preceq, d)$ be an ordered metric space and $F, G : X^2 \to X$ be two mappings. We say that (F, G) is a generalized Meir-Keeler type ψ-contraction pair if, for all $\varepsilon > 0$, there exists $\eta > 0$ such that, for all comparable pairs $(x, y), (u, v) \in X^2$,*

$$\varepsilon \leq \frac{1}{2} M(x, y, u, v) \leq \varepsilon + \eta(\varepsilon) \Rightarrow d(F(x, y), G(u, v)) < \varepsilon, \tag{1.1}$$

where

$$M(x, y, u, v) = \psi \left\{ \begin{array}{c} \delta((x, y), (u, v)), \delta((x, y), (F(x, y), F(y, x))), \\ \delta((u, v), (G(u, v), G(v, u))), \\ \frac{1}{2} \left[\begin{array}{c} \delta((x, y), (G(u, v), G(v, u))) \\ +\delta((u, v), (F(x, y), F(y, x))) \end{array} \right] \end{array} \right\}$$

with $\psi \in \Psi$.

If we take $F = G$ in the above definition, then we have:

Definition 1.6. *Let $(X, \preceq, d)$ be an ordered metric space and $F : X^2 \to X$ be a mapping. We say that F is a generalized Meir-Keeler type ψ-contraction if, for all $\varepsilon > 0$, there exists $\eta > 0$ such that, for all comparable pairs $(x, y), (u, v) \in X^2$,*

$$\varepsilon \leq \frac{1}{2} M_F(x, y, u, v) \leq \varepsilon + \eta(\varepsilon) \Rightarrow d(F(x, y), F(u, v)) < \varepsilon,$$

where

$$M_F(x,y,u,v)=\psi\left\{\begin{array}{c}\delta((x,y),(u,v)),\delta((x,y),(F(x,y),F(y,x))),\\ \delta((u,v),(F(u,v),F(v,u))),\\ \frac{1}{2}\left[\begin{array}{c}\delta((x,y),(F(u,v),F(v,u)))\\ +\delta((u,v),(F(x,y),F(y,x)))\end{array}\right]\end{array}\right\}$$

with $\psi\in\Psi$.

Remark 1.1. *Let* $(X,\preceq,d)$ *be an ordered metric space and let* $F,G:X^2\to X$ *be two given mappings. If* (F,G) *is a generalized Meir-Keeler type* ψ*-contraction pair, then*

$$d(F(x,y),G(u,v))<\frac{1}{2}M(x,y,u,v),$$

for all comparable pairs $(x,y),(u,v)\in X^2$ *when* $M(x,y,u,v)>0$. *Also, if* $M(x,y,u,v)=0$, *then* $d(F(x,y),G(u,v))=0$, *that is,*

$$d(F(x,y),G(u,v))\leq\frac{1}{2}M(x,y,u,v),$$

for all comparable pairs $(x,y),(u,v)\in X^2$.

2. Main Results

Our main result is stated as follows:

Theorem 2.1. *Let* $(X,\preceq,d)$ *be a complete ordered metric space,* $F,G:X^2\to X$ *be two weakly increasing mappings with respect to* $\preceq$ *and* (F,G) *be a generalized Meir-Keeler type* ψ*-contraction pair. If either* F *or* G *is continuous, then* F *and* G *have a coupled common fixed point.*

Proof. Let $x_0,y_0\in X$. Define two sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$x_{2n+1}=F(x_{2n},y_{2n}),\quad x_{2n+2}=G(x_{2n+1},y_{2n+1}),$$

and

$$y_{2n+1}=F(y_{2n},x_{2n}),\quad y_{2n+2}=G(y_{2n+1},x_{2n+1}),$$

for all $n\in\mathbb{N}_0$ where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Since F and G are weakly increasing, we have

$$\begin{aligned}x_1=F(x_0,y_0)&\preceq G(F(x_0,y_0),F(y_0,x_0))=G(x_1,y_1)\\ &=x_2\preceq F(G(x_1,y_1),G(y_1,x_1))=F(x_2,y_2)=x_3\preceq\cdots,\end{aligned}$$

and

$$\begin{aligned}y_1=F(y_0,x_0)&\preceq G(F(y_0,x_0),F(x_0,y_0))=G(y_1,x_1)\\ &=y_2\preceq F(G(y_1,x_1),G(x_1,y_1))=F(y_2,x_2)=y_3\preceq\cdots,\end{aligned}$$

so that the sequences $\{x_n\}$ and $\{y_n\}$ are nondecreasing.
Step 1. Now, we show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $(X,\preceq,d)$.
Case 1. If for some $n\in\mathbb{N}_0$, $x_n=x_{n+1}$ and $y_n=y_{n+1}$, then $x_{n+1}=x_{n+2}$ and $y_{n+1}=y_{n+2}$. If not, then for $n=2m$ where $m\in\mathbb{N}$, by Remark 1.1, we have

$$\begin{aligned}d(x_{n+1},x_{n+2})&=d(x_{2m+1},x_{2m+2})\\ &=d(F(x_{2m},y_{2m}),G(x_{2m+1},y_{2m+1}))\\ &<\frac{1}{2}M(x_{2m},y_{2m},x_{2m+1},y_{2m+1}),\end{aligned}$$

where

$$
\begin{aligned}
& M\left(x_{2m}, y_{2m}, x_{2m+1}, y_{2m+1}\right) \\
& \quad = \psi\left\{\begin{array}{c}
\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right), \\
\delta\left(\left(x_{2m}, y_{2m}\right),\left(F\left(x_{2m}, y_{2m}\right), F\left(y_{2m}, x_{2m}\right)\right)\right), \\
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(G\left(x_{2m+1}, y_{2m+1}\right), G\left(y_{2m+1}, x_{2m+1}\right)\right)\right), \\
\frac{1}{2}\left[\begin{array}{c}
\delta\left(\left(x_{2m}, y_{2m}\right),\left(G\left(x_{2m+1}, y_{2m+1}\right), G\left(y_{2m+1}, x_{2m+1}\right)\right)\right) \\
+\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(F\left(x_{2m}, y_{2m}\right), F\left(y_{2m}, x_{2m}\right)\right)\right)
\end{array}\right]
\end{array}\right\} \\
& \quad = \psi\left\{\begin{array}{c}
\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right), \delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right), \\
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \\
\frac{1}{2}\left[\begin{array}{c}
\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+2}, y_{2m+2}\right)\right) \\
+\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+1}, y_{2m+1}\right)\right)
\end{array}\right]
\end{array}\right\} \\
& \quad = \psi\left\{0,0, \delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \frac{1}{2}\left[\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+2}, y_{2m+2}\right)\right)\right]\right\}.
\end{aligned}
$$

Since ψ is nondecreasing, we deduce

$$
\begin{aligned}
& M\left(x_{2m}, y_{2m}, x_{2m+1}, y_{2m+1}\right) \\
& \quad \leq \psi\left\{\begin{array}{c}
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \\
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \\
\frac{1}{2}\left[\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right)+\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right)\right]
\end{array}\right\} \\
& \quad \leq \psi\left\{\begin{array}{c}
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \\
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right)
\end{array}\right\} \\
& \quad \leq \delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right)=\delta\left(\left(x_{n+1}, y_{n+1}\right),\left(x_{n+2}, y_{n+2}\right)\right).
\end{aligned}
$$

Hence, it follows that

$$
d\left(x_{n+1}, x_{n+2}\right)<\frac{1}{2} \delta\left(\left(x_{n+1}, y_{n+1}\right),\left(x_{n+2}, y_{n+2}\right)\right). \tag{2.1}
$$

Similarly, we can also show that

$$
d\left(y_{n+1}, y_{n+2}\right)<\frac{1}{2} \delta\left(\left(x_{n+1}, y_{n+1}\right),\left(x_{n+2}, y_{n+2}\right)\right). \tag{2.2}
$$

Thus, from (2.1) and (2.2)

$$
\delta\left(\left(x_{n+1}, y_{n+1}\right),\left(x_{n+2}, y_{n+2}\right)\right)<\delta\left(\left(x_{n+1}, y_{n+1}\right),\left(x_{n+2}, y_{n+2}\right)\right),
$$

which is a contradiction. Hence we must have $x_{n+1}=x_{n+2}$ and $y_{n+1}=y_{n+2}$ when n is even. By similar arguments we can show that this equality holds also when n is odd. Therefore, in any case for all those n for which $x_n=x_{n+1}$ and $y_n=y_{n+1}$ holds, we always obtain $x_{n+1}=x_{n+2}$ and $y_{n+1}=y_{n+2}$. Repeating above process inductively, one obtains $x_n=x_{n+k}$ and $y_n=y_{n+k}$ for all $k \in \mathbb{N}$. Hence $\{x_n\}$ and $\{y_n\}$ are constant sequences so that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $(X, \preceq, d)$.

Case 2. Suppose that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}_0$. Then, for $n=2m+1$, using Remark 1.1, we have

$$
\begin{aligned}
d\left(x_n, x_{n+1}\right) & =d\left(x_{2m+1}, x_{2m+2}\right) \\
& =d\left(F(x_{2m}, y_{2m}), G\left(x_{2m+1}, y_{2m+1}\right)\right) \\
& <\frac{1}{2} M\left(x_{2m}, y_{2m}, x_{2m+1}, y_{2m+1}\right),
\end{aligned} \tag{2.3}
$$

where

$$
\begin{aligned}
& M\left(x_{2m}, y_{2m}, x_{2m+1}, y_{2m+1}\right) \\
& \quad = \psi\left\{\begin{array}{c}
\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right), \delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right), \\
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \\
\frac{1}{2}\left[\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+2}, y_{2m+2}\right)\right)+\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+1}, y_{2m+1}\right)\right)\right]
\end{array}\right\}.
\end{aligned}
$$

Since,

$$
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+1}, y_{2m+1}\right)\right)=d\left(x_{2m+1}, x_{2m+1}\right)+d\left(y_{2m+1}, y_{2m+1}\right)=0,
$$

and

$$\begin{aligned}
&\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+2}, y_{2m+2}\right)\right) \\
&\quad=d\left(x_{2m}, x_{2m+2}\right)+d\left(y_{2m}, y_{2m+2}\right) \\
&\quad\leq d\left(x_{2m}, x_{2m+1}\right)+d\left(x_{2m+1}, x_{2m+2}\right)+d\left(y_{2m}, y_{2m+1}\right)+d\left(y_{2m+1}, y_{2m+2}\right) \\
&\quad=\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right)+\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right),
\end{aligned}$$

so we get

$$\begin{aligned}
&M\left(x_{2m}, y_{2m}, x_{2m+1}, y_{2m+1}\right) \\
&\quad\leq \psi\left\{\begin{array}{c}
\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right), \delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right), \\
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \\
\frac{1}{2}\left[\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right)+\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right)\right]
\end{array}\right\}.
\end{aligned} \tag{2.4}$$

If $\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right) \leq \delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right)$ for some m, since ψ is nondecreasing, we obtain

$$\begin{aligned}
&M\left(x_{n-1}, y_{n-1}, x_n, y_n\right) \\
&\quad=M\left(x_{2m}, y_{2m}, x_{2m+1}, y_{2m+1}\right) \\
&\quad\leq \psi\left\{\begin{array}{c}
\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right), \delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right), \\
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \\
\frac{1}{2}\left[\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right)+\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right)\right]
\end{array}\right\} \\
&\quad\leq \psi\left\{\begin{array}{c}
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \\
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right), \delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right)
\end{array}\right\} \\
&\quad\leq \delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right)=\delta\left(\left(x_n, y_n\right),\left(x_{n+1}, y_{n+1}\right)\right),
\end{aligned}$$

and so from (2.3)

$$d\left(x_n, x_{n+1}\right)<\frac{1}{2} \delta\left(\left(x_n, y_n\right),\left(x_{n+1}, y_{n+1}\right)\right). \tag{2.5}$$

Similarly, we can show that

$$d\left(y_n, y_{n+1}\right)<\frac{1}{2} \delta\left(\left(x_n, y_n\right),\left(x_{n+1}, y_{n+1}\right)\right). \tag{2.6}$$

Thus, from (2.5) and (2.6), we get

$$\delta\left(\left(x_n, y_n\right),\left(x_{n+1}, y_{n+1}\right)\right)<\delta\left(\left(x_n, y_n\right),\left(x_{n+1}, y_{n+1}\right)\right),$$

which is a contradiction. Hence, it must be

$$\begin{aligned}
\delta\left(\left(x_{2m+1}, y_{2m+1}\right),\left(x_{2m+2}, y_{2m+2}\right)\right) &=\delta\left(\left(x_n, y_n\right),\left(x_{n+1}, y_{n+1}\right)\right) \\
&<\delta\left(\left(x_{n-1}, y_{n-1}\right),\left(x_n, y_n\right)\right) \\
&=\delta\left(\left(x_{2m}, y_{2m}\right),\left(x_{2m+1}, y_{2m+1}\right)\right),
\end{aligned}$$

for all $n \in \mathbb{N}_0$. Set $\delta_n:=\left\{\delta\left(\left(x_n, y_n\right),\left(x_{n+1}, y_{n+1}\right)\right)\right\}$, then the sequence $\left\{\delta_n\right\}$ is decreasing and bounded below. Thus, there exists $r \geq 0$ such that $\lim_{n \rightarrow \infty} \delta_n=r$. Notice that

$$r=\inf \left\{\delta_n: n \in \mathbb{N}_0\right\}. \tag{2.7}$$

We now prove that $r=0$. If not, then by (2.4), we deduce that

$$\lim_{n \rightarrow \infty} M\left(x_{n-1}, y_{n-1}, x_n, y_n\right)=r.$$

Then there exists a positive integer p such that

$$\varepsilon \leq \frac{1}{2} M\left(x_{p-1}, y_{p-1}, x_p, y_p\right)<\varepsilon+\eta(\varepsilon),$$

where $\varepsilon=r/2$. Owing to the condition (1.1), we have

$$d\left(F(x_{p-1}, y_{p-1}), G\left(x_p, y_p\right)\right)<\varepsilon,$$

which implies

$$d\left(x_p, x_{p+1}\right)<\varepsilon.$$

Similarly, we can obtain that

$$d\left(y_p, y_{p+1}\right) < \varepsilon.$$

Summing the two foregoing inequalities, we get

$$\delta\left(\left(x_p, y_p\right), \left(x_{p+1}, y_{p+1}\right)\right) < 2\varepsilon = r,$$

which contradicts (2.7) for $n = p$. Thus, $\varepsilon = r/2 = 0$, that is,

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \left[d\left(x_n, x_{n+1}\right) + d\left(y_n, y_{n+1}\right)\right] = 0. \tag{2.8}$$

Now, we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. It is sufficient to show that $\{x_{2n}\}$ and $\{y_{2n}\}$ are Cauchy sequences in (X, d). Suppose, to the contrary, that at least one of $\{x_{2n}\}$ or $\{y_{2n}\}$ is not Cauchy sequence. Then, there exists an $\varepsilon > 0$ for which we can find subsequences $\{x_{2m_k}\}, \{x_{2n_k}\}$ of $\{x_{2n}\}$ and $\{y_{2m_k}\}, \{y_{2n_k}\}$ of $\{y_{2n}\}$ such that n_k is the smallest index for which $n_k > m_k > k$ and

$$d(x_{2n_k}, x_{2m_k}) + d(y_{2n_k}, y_{2m_k}) \geq \varepsilon \ \text{ and } \ d(x_{2n_{k-1}}, x_{2m_k}) + d(y_{2n_{k-1}}, y_{2m_k}) < \varepsilon. \tag{2.9}$$

Using the triangular inequality and (2.9), we get

$$\begin{aligned}
\varepsilon &\leq d(x_{2n_k}, x_{2m_k}) + d(y_{2n_k}, y_{2m_k}) \\
&\leq d(x_{2m_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k}) + d(y_{2m_k}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}) \\
&< \varepsilon + \delta_{2n_k-1}.
\end{aligned}$$

Taking $k \to \infty$ in the above inequality and using (2.8), we deduce

$$\lim_{k\to\infty} [d(x_{2n_k}, x_{2m_k}) + d(y_{2n_k}, y_{2m_k})] = \varepsilon. \tag{2.10}$$

Again, by the triangle inequality, we get

$$\begin{aligned}
d(x_{2n_k}, x_{2m_k}) + d(y_{2n_k}, y_{2m_k}) &\leq d(x_{2n_k}, x_{2n_k+1}) + d(x_{2n_k+1}, x_{2m_k-1}) + d(x_{2m_k-1}, x_{2m_k}) \\
&\quad + d(y_{2n_k}, y_{2n_k+1}) + d(y_{2n_k+1}, y_{2m_k-1}) + d(y_{2m_k-1}, y_{2m_k}) \\
&\leq \delta_{2n_k} + \delta_{2m_k-1} + d(x_{2n_k+1}, x_{2m_k}) + d(x_{2m_k}, x_{2m_k-1}) \\
&\quad + d(y_{2n_k+1}, y_{2m_k}) + d(y_{2m_k}, y_{2m_k-1}) \\
&\leq \delta_{2n_k} + 2\delta_{2m_k-1} + d(x_{2n_k+1}, x_{2n_k}) + d(x_{2n_k}, x_{2m_k}) \\
&\quad + d(y_{2n_k+1}, y_{2n_k}) + d(y_{2n_k}, y_{2m_k}) \\
&= 2\delta_{2n_k} + 2\delta_{2m_k-1} + d(x_{2n_k}, x_{2m_k}) + d(y_{2n_k}, y_{2m_k}).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality besides using (2.8) and (2.10), we have

$$\begin{array}{l}
\lim_{k\to\infty} [d(x_{2n_k+1}, x_{2m_k-1}) + d(y_{2n_k+1}, y_{2m_k-1})] = \varepsilon, \ \text{and} \\
\lim_{k\to\infty} [d(x_{2n_k+1}, x_{2m_k}) + d(y_{2n_k+1}, y_{2m_k})] = \varepsilon.
\end{array} \tag{2.11}$$

On the other hand, we also obtain

$$\begin{aligned}
d(x_{2n_k}, x_{2m_k-1}) + d(y_{2n_k}, y_{2m_k-1}) &\leq d(x_{2n_k}, x_{2m_k}) + d(x_{2m_k}, x_{2m_k-1}) \\
&\quad + d(y_{2n_k}, y_{2m_k}) + d(y_{2m_k}, y_{2m_k-1}) \\
&= d(x_{2n_k}, x_{2m_k}) + d(y_{2n_k}, y_{2m_k}) + \delta_{2m_k-1},
\end{aligned}$$

which implies

$$\lim_{k\to\infty} [d(x_{2n_k}, x_{2m_k-1}) + d(y_{2n_k}, y_{2m_k-1})] \leq \varepsilon. \tag{2.12}$$

Since $(x_{2n_k}, y_{2n_k}) \succeq (x_{2m_k-1}, y_{2m_k-1})$ for $n_k > m_k$, using Remark 1.1, we have

$$\begin{aligned}
d\left(x_{2n_k+1}, x_{2m_k}\right) &= d\left(F(x_{2n_k}, y_{2n_k}), G\left(x_{2m_k-1}, y_{2m_k-1}\right)\right) \\
&< \frac{1}{2} M\left(x_{2n_k}, y_{2n_k}, x_{2m_k-1}, y_{2m_k-1}\right),
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
&M\left(x_{2n_k}, y_{2n_k}, x_{2m_k-1}, y_{2m_k-1}\right) \\
&= \psi\left\{\begin{array}{c}
\delta\left(\left(x_{2n_k}, y_{2n_k}\right), \left(x_{2m_k-1}, y_{2m_k-1}\right)\right), \\
\delta\left(\left(x_{2n_k}, y_{2n_k}\right), \left(F(x_{2n_k}, y_{2n_k}), F(y_{2n_k}, x_{2n_k})\right)\right), \\
\delta\left(\left(x_{2m_k-1}, y_{2m_k-1}\right), \left(G\left(x_{2m_k-1}, y_{2m_k-1}\right), G\left(y_{2m_k-1}, x_{2m_k-1}\right)\right)\right), \\
\frac{1}{2}\left[\begin{array}{c}
\delta\left(\left(x_{2n_k}, y_{2n_k}\right), \left(G\left(x_{2m_k-1}, y_{2m_k-1}\right), G\left(y_{2m_k-1}, x_{2m_k-1}\right)\right)\right) \\
+\delta\left(\left(x_{2m_k-1}, y_{2m_k-1}\right), \left(F(x_{2n_k}, y_{2n_k}), F(y_{2n_k}, x_{2n_k})\right)\right)
\end{array}\right]
\end{array}\right\} \\
&= \psi\left\{\begin{array}{c}
\delta\left(\left(x_{2n_k}, y_{2n_k}\right), \left(x_{2m_k-1}, y_{2m_k-1}\right)\right), \delta\left(\left(x_{2n_k}, y_{2n_k}\right), \left(x_{2n_k+1}, y_{2n_k+1}\right)\right), \\
\delta\left(\left(x_{2m_k-1}, y_{2m_k-1}\right), \left(x_{2m_k}, y_{2m_k}\right)\right), \\
\frac{1}{2}\left[\delta\left(\left(x_{2n_k}, y_{2n_k}\right), \left(x_{2m_k}, y_{2m_k}\right)\right) + \delta\left(\left(x_{2m_k-1}, y_{2m_k-1}\right), \left(x_{2n_k+1}, y_{2n_k+1}\right)\right)\right]
\end{array}\right\}.
\end{aligned}$$

By a similar method, we can also show that

$$d\left(y_{2n_k+1}, y_{2m_k}\right) < \frac{1}{2} M\left(x_{2n_k}, y_{2n_k}, x_{2m_k-1}, y_{2m_k-1}\right). \tag{2.14}$$

Summing the inequalities (2.13) and (2.14), we get

$$d\left(x_{2n_k+1}, x_{2m_k}\right) + d\left(y_{2n_k+1}, y_{2m_k}\right) < M\left(x_{2n_k}, y_{2n_k}, x_{2m_k-1}, y_{2m_k-1}\right).$$

Now, using (2.8), (2.10), (2.11) and (2.12) as $k \to \infty$ in the above inequality, we deduce

$$\begin{aligned}
\varepsilon &< \lim_{k\to\infty} M\left(x_{2n_k}, y_{2n_k}, x_{2m_k-1}, y_{2m_k-1}\right) \\
&= \psi\{\lim_{k\to\infty} \delta\left(\left(x_{2n_k}, y_{2n_k}\right), \left(x_{2m_k-1}, y_{2m_k-1}\right)\right), 0, 0, \varepsilon\} \\
&\le \psi\left\{\varepsilon, 0, 0, \varepsilon\right\} \le \varepsilon,
\end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X.
Step 2. Now, we prove the existence of coupled common fixed point of F and G.
Owing to the completeness of (X, d), there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y. \tag{2.15}$$

Without loss of generality, we assume that F is continuous. Now, we have

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} F(x_{2n}, y_{2n}) = F\left(\lim_{n\to\infty} x_{2n}, \lim_{n\to\infty} y_{2n}\right) = F\left(x, y\right).$$

and

$$y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} F(y_{2n}, x_{2n}) = F\left(\lim_{n\to\infty} y_{2n}, \lim_{n\to\infty} x_{2n}\right) = F\left(y, x\right).$$

We now assert that $d\left(x, G\left(x, y\right)\right) = d\left(y, G\left(y, x\right)\right) = 0$. To establish the claim, assume that $d\left(x, G\left(x, y\right)\right) > 0$ and $d\left(y, G\left(y, x\right)\right) > 0$. Since $(x, y) \in X^2$ is comparable to its own, making use of Remark 1.1, we obtain

$$d\left(x, G\left(x, y\right)\right) = d\left(F\left(x, y\right), G\left(x, y\right)\right) < \frac{1}{2} M\left(x, y, x, y\right), \tag{2.16}$$

where

$$\begin{aligned}
M\left(x, y, x, y\right) &= \psi\left\{\begin{array}{c}
\delta\left(\left(x, y\right), \left(x, y\right)\right), \delta\left(\left(x, y\right), \left(F(x, y), F(y, x)\right)\right), \\
\delta\left(\left(x, y\right), \left(G\left(x, y\right), G\left(y, x\right)\right)\right), \\
\frac{1}{2}\left[\delta\left(\left(x, y\right), \left(G\left(x, y\right), G\left(y, x\right)\right)\right) + \delta\left(\left(x, y\right), \left(F(x, y), F(y, x)\right)\right)\right]
\end{array}\right\} \\
&= \psi\left\{0, 0, \delta\left(\left(x, y\right), \left(G\left(x, y\right), G\left(y, x\right)\right)\right), \frac{1}{2}\delta\left(\left(x, y\right), \left(G\left(x, y\right), G\left(y, x\right)\right)\right)\right\} \\
&\le \delta\left(\left(x, y\right), \left(G\left(x, y\right), G\left(y, x\right)\right)\right).
\end{aligned}$$

Similarly, we have

$$d\left(y, G\left(y, x\right)\right) < \frac{1}{2} M\left(x, y, x, y\right). \tag{2.17}$$

Thus, it follows from (2.16) and (2.17) that

$$\begin{aligned} d(x, G(x,y)) + d(y, G(y,x)) &< M(x,y,x,y) \\ &\leq \delta((x,y),(G(x,y),G(y,x))) \\ &= d(x, G(x,y)) + d(y, G(y,x)), \end{aligned}$$

which implies $d(x, G(x,y)) = d(y, G(y,x)) = 0$. Hence, $x = F(x,y) = G(x,y)$ and $y = F(y,x) = G(y,x)$. □

Now, we furnish the following example which illustrates the results of Theorem 2.1.

Example 2.1. *Let $X = [0,1]$ be equipped with the usual metric and the partial order defined by*

$$x \preceq y \Longleftrightarrow y \leq x.$$

Define the function $\psi : \mathbb{R}^{+^4} \to \mathbb{R}^+$ by $\psi(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}$ and the mappings $F, G : X^2 \to X$ by $F(x,y) = \frac{x+y}{7}$ and $G(x,y) = \frac{x+y}{6}$. Then, it is easy to see that F and G are weakly increasing with respect to $\preceq$.

Also, (F,G) is a generalized Meir-Keeler type ψ-contraction. Indeed, for all comparable $(x,y),(u,v) \in X^2$

$$\begin{aligned} d(F(x,y), G(u,v)) &= \left| \frac{x+y}{7} - \frac{u+v}{6} \right| \\ &\leq \frac{1}{7}(|x-u| + |y-v|) \\ &= \frac{1}{7}\delta((x,y),(u,v)) \leq \frac{1}{7} M(x,y,u,v) \\ &< \frac{1}{7} \cdot 2(\varepsilon + \eta(\varepsilon)) < \varepsilon, \end{aligned}$$

which holds if we choose $\eta(\varepsilon) < \frac{5}{2}\varepsilon$. Thus, it can easily see that all the hypotheses of Theorem 2.1 are fulfilled. Therefore, F and G have a coupled common fixed point which is $(0,0)$.

Definition 2.1. *Let $(X, \preceq, d)$ be an ordered metric space. We say that $(X, \preceq, d)$ is regular if for non-decreasing sequence $\{x_n\}$ with $d(x_n, x) \to 0$ implies that $x_n \preceq x$ for all n.*

In our next theorem, we replace the continuity of F or G in Theorem 2.1 with the regularity of $(X, \preceq, d)$.

Theorem 2.2. *Let $(X, \preceq, d)$ be a complete ordered metric space, $F, G : X^2 \to X$ be two weakly increasing mappings with respect to $\preceq$ and (F,G) be a generalized Meir-Keeler type ψ-contraction pair. If $(X, \preceq, d)$ is regular, then F and G have a coupled common fixed point.*

Proof. We define sequences $\{x_n\}$ and $\{y_n\}$ as in Theorem 2.1. Proceeding on the lines of the proof of Theorem 2.1, we can show that the sequences $\{x_n\}$ and $\{y_n\}$ are non-decreasing and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then, since $(X, \preceq, d)$ is regular, we conclude that (x_n, y_n) is comparable to (x,y) for all $n \in \mathbb{N}_0$. Now, by Remark 1.1, we obtain

$$d(x_{2n+1}, G(x,y)) = d(F(x_{2n}, y_{2n}), G(x,y)) < \frac{1}{2} M(x_{2n}, y_{2n}, x, y), \tag{2.18}$$

where

$$\begin{aligned} &M(x_{2n}, y_{2n}, x, y) \\ &= \psi \left\{ \begin{array}{c} \delta((x_{2n}, y_{2n}), (x,y)), \delta((x_{2n}, y_{2n}), (F(x_{2n}, y_{2n}), F(y_{2n}, x_{2n}))), \\ \delta((x,y), (G(x,y), G(y,x))), \\ \frac{1}{2}\left[\begin{array}{c} \delta((x_{2n}, y_{2n}), (G(x,y), G(y,x))) \\ +\delta((x,y), (F(x_{2n}, y_{2n}), F(y_{2n}, x_{2n}))) \end{array} \right] \end{array} \right\} \\ &= \psi \left\{ \begin{array}{c} \delta((x_{2n}, y_{2n}), (x,y)), \delta((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})), \\ \delta((x,y), (G(x,y), G(y,x))), \\ \frac{1}{2}[\delta((x_{2n}, y_{2n}), (G(x,y), G(y,x))) + \delta((x,y), (x_{2n+1}, y_{2n+1}))] \end{array} \right\}. \end{aligned}$$

Similarly, we have

$$d\left(y_{2n+1}, G\left(y, x\right)\right) = d\left(F(y_{2n}, x_{2n}), G\left(y, x\right)\right) < \frac{1}{2} M\left(x_{2n}, y_{2n}, x, y\right). \tag{2.19}$$

Thus it follows from (2.18) and (2.19) that

$$d\left(x_{2n+1}, G\left(x, y\right)\right) + d\left(y_{2n+1}, G\left(y, x\right)\right) < M\left(x_{2n}, y_{2n}, x, y\right).$$

Letting $n \to \infty$ in the above inequality, we get

$$\begin{aligned} d\left(x, G\left(x, y\right)\right) + d\left(y, G\left(y, x\right)\right) &< \psi \left\{ \begin{array}{c} 0, 0, \delta\left(\left(x, y\right), \left(G\left(x, y\right), G\left(y, x\right)\right)\right), \\ \frac{1}{2}\delta\left(\left(x, y\right), \left(G\left(x, y\right), G\left(y, x\right)\right)\right) \end{array} \right\} \\ &\leq \delta\left(\left(x, y\right), \left(G\left(x, y\right), G\left(y, x\right)\right)\right) \\ &= d\left(x, G\left(x, y\right)\right) + d\left(y, G\left(y, x\right)\right), \end{aligned}$$

which implies $d(x, G\left(x, y\right)) = 0$ and $d\left(y, G\left(y, x\right)\right) = 0$ that is, $x = G\left(x, y\right)$ and $y = G\left(y, x\right).$

Since $(x, y) \in X^2$ is comparable to its own, using of Remark 1.1, we obtain

$$d\left(F\left(x, y\right), x\right) = d\left(F\left(x, y\right), G\left(x, y\right)\right) < \frac{1}{2} M\left(x, y, x, y\right), \tag{2.20}$$

where

$$\begin{aligned} M\left(x, y, x, y\right) &= \psi \left\{ \begin{array}{c} \delta\left(\left(x, y\right), \left(x, y\right)\right), \delta\left(\left(x, y\right), \left(F(x, y), F(y, x)\right)\right), \\ \delta\left(\left(x, y\right), \left(G\left(x, y\right), G\left(y, x\right)\right)\right), \\ \frac{1}{2}\left[\delta\left(\left(x, y\right), \left(G\left(x, y\right), G\left(y, x\right)\right)\right) + \delta\left(\left(x, y\right), \left(F(x, y), F(y, x)\right)\right)\right] \end{array} \right\} \\ &= \psi \left\{ 0, \delta\left(\left(x, y\right), \left(F(x, y), F(y, x)\right)\right), 0, \frac{1}{2}\delta\left(\left(x, y\right), \left(F(x, y), F(y, x)\right)\right) \right\} \\ &\leq \delta\left(\left(x, y\right), \left(F(x, y), F(y, x)\right)\right). \end{aligned}$$

Similarly, we have

$$d\left(F\left(y, x\right), y\right) < \frac{1}{2} M\left(x, y, x, y\right). \tag{2.21}$$

Thus, it follows from (2.20) and (2.21) that

$$\begin{aligned} d\left(F\left(x, y\right), x\right) + d\left(F\left(y, x\right), y\right) &< M\left(x, y, x, y\right) \\ &\leq \delta\left(\left(x, y\right), \left(F(x, y), F(y, x)\right)\right) \\ &= d\left(x, F\left(x, y\right)\right) + d\left(y, F\left(y, x\right)\right), \end{aligned}$$

which implies $d\left(x, F\left(x, y\right)\right) = d\left(y, F\left(y, x\right)\right) = 0$. Therefore, $x = F\left(x, y\right) = G\left(x, y\right)$ and $y = F\left(y, x\right) = G\left(y, x\right).$ □

Definition 2.2. *Let $(X, \preceq, d)$ be an ordered metric space and $F, G : X^2 \to X$ be two mappings. We say that (F, G) is a generalized Meir-Keeler type contraction if, for all $\varepsilon > 0$, there exists $\eta > 0$ such that, for all comparable $(x, y), (u, v) \in X^2$,*

$$\varepsilon \leq \frac{1}{2} M_{\max}\left(x, y, u, v\right) \leq \varepsilon + \eta\left(\varepsilon\right) \Rightarrow d\left(F\left(x, y\right), G\left(u, v\right)\right) < \varepsilon, \tag{2.22}$$

where

$$M_{\max}\left(x, y, u, v\right) = \max \left\{ \begin{array}{c} \delta\left(\left(x, y\right), \left(u, v\right)\right), \delta\left(\left(x, y\right), \left(F\left(x, y\right), F\left(y, x\right)\right)\right), \\ \delta(\left(u, v\right), \left(G\left(u, v\right), G\left(v, u\right)\right)), \\ \frac{1}{2}\left[\begin{array}{c} \delta\left(\left(x, y\right), \left(G\left(u, v\right), G\left(v, u\right)\right)\right) \\ +\delta\left(u, v\right), \left(F\left(x, y\right), F\left(y, x\right)\right) \end{array} \right] \end{array} \right\}.$$

If we take $\psi\left(t_1, t_2, t_3, t_4\right) = \max\left\{t_1, t_2, t_3, t_4\right\}$ in Theorem 2.1, we have the following result.

Corollary 2.1. *Let $(X, \preceq, d)$ be a complete ordered metric space, $F, G : X^2 \to X$ be weakly increasing mappings with respect to $\preceq$ and (F, G) be a generalized Meir-Keeler type contraction. Assume that the following conditions are satisfied:*

(*a*) *F (or G) is continuous or*
(*b*) *$(X, \preceq, d)$ is regular.*

Then F and G have a coupled common fixed point.

Proposition 2.1. *Let $(X, \preceq, d)$ be an ordered metric space and let $F, G : X^2 \to X$ be two given mappings. If the following contraction is satisfied, then (F, G) is a generalized Meir-Keeler type contraction:*

$$d\left(F\left(x,y\right),G\left(u,v\right)\right) \leq \frac{k}{2} M_{\max}\left(x,y,u,v\right), \qquad k \in [0,1).$$

Proof. Assume that the above inequality holds. Then, for all $\varepsilon > 0$, we can easily show that (2.22) is satisfied with $\eta\left(\varepsilon\right) = \left(\frac{1}{k} - 1\right)\varepsilon$. □

Definition 2.3. *Let $(X, \preceq)$ be an ordered set and $F : X^2 \to X$. We say that F is nondecreasing if, for any $x, y \in X$,*

$$x_1, x_2 \in X, \qquad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \qquad y_1 \preceq y_2 \Rightarrow F(x, y_1) \preceq F(x, y_2).$$

If we choose $F = G$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $(X, \preceq, d)$ be a complete ordered metric space, $F : X^2 \to X$ be a nondecreasing mapping and F be a generalized Meir-Keeler type ψ-contraction. Assume that the following conditions hold:*

- (a) *F is continuous or*
- (b) *$(X, \preceq, d)$ is regular.*

Then F has a coupled fixed point.

We denote by Φ the family of all functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

- (ϕ_1) ϕ is nondecreasing and right continuous;
- (ϕ_2) $\phi\left(0\right) = 0$ and $\phi\left(t\right) > 0$ for any $t > 0$.

Theorem 2.3. *Let $(X, \preceq, d)$ be a complete ordered metric space and $F, G : X^2 \to X$ be two weakly increasing mappings with respect to $\preceq$. Assume that, for all $\varepsilon > 0$, there exists $\eta > 0$ such that, for all comparable $\left(x, y\right), \left(u, v\right) \in X^2$,*

$$\varepsilon \leq \phi\left(\frac{1}{2} M\left(x,y,u,v\right)\right) \leq \varepsilon + \eta\left(\varepsilon\right) \Rightarrow \phi\left(d\left(F\left(x,y\right),G\left(u,v\right)\right)\right) < \varepsilon, \tag{2.23}$$

where $\phi \in \Phi$. Then (F, G) is a generalized Meir-Keeler type ψ-contraction.

Proof. Fix $\varepsilon > 0$. Since $\phi\left(\varepsilon\right) > 0$, there exists $\gamma > 0$ such that, for all comparable $\left(u, v\right), \left(w, z\right) \in X^2$,

$$\phi\left(\varepsilon\right) \leq \phi\left(\frac{1}{2} M\left(u,v,w,z\right)\right) \leq \phi\left(\varepsilon\right) + \gamma \Rightarrow \phi\left(d\left(F\left(x,y\right),G\left(u,v\right)\right)\right) < \phi\left(\varepsilon\right). \tag{2.24}$$

Due to the right continuity of ϕ, there exists $\eta > 0$ such that $\phi\left(\varepsilon + \eta\right) < \phi\left(\varepsilon\right) + \gamma$. For any comparable $\left(x, y\right), \left(u, v\right) \in X^2$, such that

$$\varepsilon \leq \frac{1}{2} M\left(x,y,u,v\right) \leq \varepsilon + \eta.$$

Since ϕ is nondecreasing, we have:

$$\phi\left(\varepsilon\right) \leq \phi\left(\frac{1}{2} M\left(x,y,u,v\right)\right) < \phi\left(\varepsilon + \eta\right) < \phi\left(\varepsilon\right) + \gamma,$$

By (2.24), we obtain $\phi\left(d\left(F\left(x,y\right),G\left(u,v\right)\right)\right) < \phi\left(\varepsilon\right)$ which implies $d\left(F\left(x,y\right),G\left(u,v\right)\right) < \varepsilon$, as ϕ is nondecreasing. This completes the proof. □

Corollary 2.3. *Let $(X, \preceq, d)$ be a complete ordered metric space and $F, G : X^2 \to X$ be two weakly increasing mappings with respect to $\preceq$. Assume that, for all $\varepsilon > 0$, there exists $\eta > 0$ such that, for every comparable pair $\left(x, y\right), \left(u, v\right) \in X^2$,*

$$\varepsilon \leq \int_0^{\frac{1}{2} M(x,y,u,v)} \theta\left(t\right) dt \leq \varepsilon + \eta\left(\varepsilon\right) \Rightarrow \int_0^{d(F(x,y),G(u,v))} \theta\left(t\right) dt < \varepsilon,$$

where θ is a locally integrable function from $\mathbb{R}^+$ into itself satisfying $\int_0^s \theta\left(t\right) dt > 0$ for all $s > 0$. Suppose also that the following conditions are satisfied:

(a) *F (or G) is continuous or*
(b) $(X, \preceq, d)$ *is regular.*

Then F and G have a coupled common fixed point.

Corollary 2.4. *Let $(X, \preceq, d)$ be a complete ordered metric space and $F, G : X^2 \to X$ be two weakly increasing mappings with respect to $\preceq$. Assume that, for all comparable $(x, y), (u, v) \in X^2$,*

$$\int_0^{d(F(x,y),G(u,v))} \theta(t)\,dt \le k \int_0^{\frac{1}{2}M(x,y,u,v)} \theta(t)\,dt,$$

where $k \in (0, 1)$ and the function θ is defined as in Corollary 2.3. Suppose also that the following conditions are satisfied:

(a) *F (or G) is continuous or*
(b) $(X, \preceq, d)$ *is regular.*

Then F and G have a coupled common fixed point.

Proof. For any $\varepsilon > 0$, choosing $\eta(\varepsilon) = \left(\frac{1}{k} - 1\right)\varepsilon$ and applying Corollary 2.3, the desired result is obtained. □

3. An Application

Consider the following integral equations:

$$\begin{aligned} x(s) &= \int_a^b H_1(s, r, x(r), y(r))\,dr, \\ y(s) &= \int_a^b H_1(s, r, y(r), x(r))\,dr, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} x(s) &= \int_a^b H_2(s, r, x(r), y(r))\,dr, \\ y(s) &= \int_a^b H_2(s, r, y(r), x(r))\,dr, \end{aligned} \tag{3.2}$$

where $s \in I = [a, b]$, $H_1, H_2 : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $b > a \ge 0$.

In this section, we present an existence theorem for a common solution to (3.1) and (3.2) that belongs to $X := C(I, \mathbb{R})$ (the set of continuous functions defined on I) by using the obtained result in Corollary 2.1.

We consider the operators $F, G : X^2 \to X$ given by

$$F(x, y)(s) = \int_a^b H_1(s, r, x(r), y(r))\,dr, \quad x, y \in X, \quad s \in I,$$

and

$$G(x, y)(s) = \int_a^b H_2(s, r, x(r), y(r))\,dr, \quad x, y \in X, \quad s \in I.$$

Then the existence of a common solution to (3.1) and (3.2) is equivalent to the existence of a coupled common fixed point of F and G.

It is well known that X endowed with the metric d defined by $d(x, y) = sup_{s\in I}|x(s) - y(s)|$ for all $x, y \in X$, forms a complete metric space. Also, equip X with the partial order $\preceq$ given by

$$x, y \in X, \qquad x \preceq y \Leftrightarrow x(s) \le y(s), \quad \forall s \in I.$$

Recall that in [14], it is proved that $(X, \preceq, d)$ is regular.

Suppose that the following conditions hold.

(A) $H_1, H_2 : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous;

(B) for all $s, r \in I$ and $x, y \in X$, we have

$$H_1(s, r, x(r), y(r)) \le H_2\left(s, r, \int_a^b H_1(r, \tau, x(\tau), y(\tau))\,d\tau, \int_a^b H_1(r, \tau, y(\tau), x(\tau))\,d\tau\right)$$

and

$$H_2(s, r, x(r), y(r)) \le H_1\left(s, r, \int_a^b H_2(r, \tau, x(\tau), y(\tau))\,d\tau, \int_a^b H_2(r, \tau, y(\tau), x(\tau))\,d\tau\right)$$

(C) for all comparable $(x, y), (u, v) \in X^2$ and for every $s, r \in I$, we have

$$|H_1(s,r,x(r),y(r)) - H_2(s,r,u(r),v(r))|^2 \le \frac{k^2}{4}\gamma(s,r)\left[|x(r)-u(r)| + |y(r)-v(r)|\right]^2,$$

where $k \in [0,1)$ and $\gamma : I^2 \to \mathbb{R}^+$ is a continuous function satisfying $\sup_{s\in I}\int_a^b \gamma(s,r) \le 1/(b-a)$.

Theorem 3.1. *Assume that conditions (A)-(C) are satisfied. Then, integral equations (3.1) and (3.2) have a common solution in X.*

Proof. From condition (B), the mappings F and G are weakly increasing with respect to $\preceq$.

Let (x,y) is comparable to (u,v). Then, by (C), for all $s \in I$, we deduce

$$\begin{aligned}
|F(x,y)(s) - G(u,v)(s)|^2 &\le \left(\int_a^b |H_1(s,r,x(r),y(r)) - H_2(s,r,u(r),v(r))|\,dr\right)^2 \\
&\le \int_a^b 1^2 dr \int_a^b |H_1(s,r,x(r),y(r)) - H_2(s,r,u(r),v(r))|^2\,dr \\
&\le (b-a)\int_a^b \frac{k^2}{4}\gamma(s,r)\left[|x(r)-u(r)| + |y(r)-v(r)|\right]^2 dr \\
&\le \frac{k^2}{4}(b-a)\int_a^b \gamma(s,r)\left[d(x,u) + d(y,v)\right]^2 dr \\
&\le \frac{k^2}{4}(b-a)\sup_{s\in I}\left(\int_a^b \gamma(s,r)\,dr\right)\left[\delta((x,y),(u,v))\right]^2 \\
&\le \left[\frac{k}{2}\delta((x,y),(u,v))\right]^2 \le \left[\frac{k}{2}M_{\max}(x,y,u,v)\right]^2,
\end{aligned}$$

where

$$M_{\max}(x,y,u,v) = \max\left\{\begin{array}{c} \delta((x,y),(u,v)), \delta((x,y),(F(x,y),F(y,x))), \\ \delta((u,v),(G(u,v),G(v,u))), \\ \frac{1}{2}\left[\begin{array}{c} \delta((x,y),(G(u,v),G(v,u))) \\ +\delta(u,v),(F(x,y),F(y,x)) \end{array}\right] \end{array}\right\}.$$

Therefore, we obtain

$$\left(\sup_{s\in I}|F(x,y)(s) - G(u,v)(s)|\right)^2 \le \left[\frac{k}{2}M_{\max}(x,y,u,v)\right]^2,$$

and so

$$d(F(x,y),G(u,v)) \le \frac{k}{2}M_{\max}(x,y,u,v). \tag{3.3}$$

Hence, by Proposition 2.1, (F,G) is a generalized Meir-Keeler type contraction. Therefore, from Corollary 2.1, F and G have a coupled common fixed point, that is, integral equations (3.1) and (3.2) have a common solution in X. □

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