Stochastic Higher-order Finite Element Model for the Free Vibration of a Continuous Beam resting on Elastic Support with Uncertain Elastic Modulus

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Received: 28 October 2022 | Revised: 3 December 2022 | Accepted: 4 December 2022

ABSTRACT

This paper deals with a continuous beam resting on elastic support with elastic modulus derived from a random process. Governing equations of the stochastic higher-order finite element method of the free vibration of the continuous beam were derived from Hamilton's principle. The random process of elastic modulus was discretized by averaging random variables in each element. A solution for the stochastic eigenvalue problem for the free vibration of the continuous beam was obtained by using the perturbation technique, in conjunction with the finite element method. Spectral representation was used to generate a random process and employ the Monte Carlo simulation. A good agreement was obtained between the results of the first-order perturbation technique and the Monte Carlo simulation.

Keywords-SFEM; free vibration; continuous beam; random field

I. INTRODUCTION

Beams and frames are used in many engineering applications, such as civil engineering [1-3], bridge engineering, and aerospace engineering. The importance of dynamic problems has been the subject of many research studies [4-9]. In civil engineering, the structures resting on foundations are very important [10-15]. Considering continuous beams resting on elastic foundations, elastic support is a common problem. Many practical problems are idealized to continuous beams resting on elastic foundations, elastic support such as rails in the railway, and strip foundations on the soil. The nonlinear dynamic of plates on a viscoelastic Winkler foundation under harmonic moving load is solved using the multiple time scales method [16]. Subway dynamic problem under loads from the ground and underground transport is modeled using finite element software in [17]. The dynamic of sandwich beams with a viscoelastic core subjected to a moving load is investigated using the finite element method in [18]. The natural frequencies of non-uniform axially functionally graded beams were investigated using the Chebyshev collocation method in [19].

For stochastic problems, material properties, loads, and geometrical dimensions are assumed to be stochastic. Stochastic dynamics problems have been the subject of many studies [20-24]. There are several types of Stochastic Finite Element Methods (SFEM), e.g. the probabilistic finite element method [25, 26], the Spectral SFEMs (SSFEMs) using Karhunen–Loève expansion series [27], the homogeneous chaos expansion method [28] for the representation of random fields, and the SFEMs that use weighed integration techniques [29, 30]. The effect of randomness in elastic modulus on the stochastic free vibration of non-uniform beams is investigated by using an SFEM in [31]. Authors in [32] dealt with the stochastic dynamics of an infinite double beam resting on a random elastic foundation subjected to a moving load. Authors in [33] investigated a beam on a random foundation using the SFEM. The stochastic buckling of non-uniform columns was analyzed using stochastic finite elements in [34].

In the present work, the SFEM is used in the study of the free vibration of a continuous beam resting on an elastic support. The results are compared with the ones of the Monte Carlo simulation.

II. STOCHASTIC FINITE ELEMENT FORMULATION FOR CONTINUOUS BEAM RESTING ON ELASTIC SUPPORT

Consider a continuous beam resting on elastic support with length *L* as shown in Figure 1.

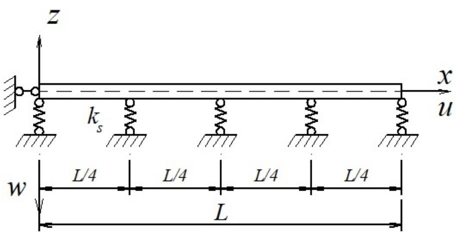

Fig. 1. Continuous beam resting on elastic support.

The displacement fields using Reddy's beam theory [35] present as formulation are:

$$u(x,z) = z\psi(x) + z^3\left(-\frac{4}{3h^2}\right)\left(\psi(x) + \frac{\partial w_0}{\partial x}\right) \quad (1)$$
$$w(x,z) = w_0(x)$$

The strain formulation is calculated from (1) as follows:

$$\varepsilon_x = \frac{\partial \psi}{\partial x}\left(z - \frac{4z^3}{3h^2}\right) - \frac{4z^3}{3h^2}\frac{\partial^2 w_0}{\partial x^2}$$
$$\gamma_{xz} = \psi\left(1 - \frac{4z^2}{h^2}\right) + \frac{\partial w_0}{\partial x}\left(1 - \frac{4z^2}{h^2}\right) \quad (2)$$

The displacement fields are approximated by an interpolation functions as follows:

$$\begin{cases} w_0(x) = N_1 w_1 + N_2\theta_1 + N_3 w_2 + N_4\theta_2 \\ \psi(x) = N_1^L\psi_1 + N_2^L\psi_2 \end{cases} \quad (3)$$

The interpolation functions are defined as follows:

$$\begin{cases} N_1 = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3} & N_3 = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} \\ N_2 = x\left(1 - 2\frac{x}{L} + \frac{x^2}{L^2}\right) & N_4 = x\left(-\frac{x}{L} + \frac{x^2}{L^2}\right) \\ N_1^L = 1 - \frac{x}{L} & N_2^L = 1 - \frac{x}{L} \end{cases} \quad (4)$$

The displacement vector of finite element is:

$$\{q\}_e = \{\psi_1 \quad w_1 \quad \theta_1 \quad \psi_2 \quad w_2 \quad \theta_2\}^T \quad (5)$$

The strain energy of beam element is given as:

$$U_e = \int_0^{L_e}\int_A \left\{ \begin{aligned} &\frac{1}{2}E\left[y\frac{\partial \phi_x}{\partial x} - \frac{4}{3h^2}y^3\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right)\right]^2 \\ &+\frac{1}{2}G\left[\left(1 - \frac{4}{h^2}\right)\left(\psi + \frac{\partial w_0}{\partial x}\right)\right]^2 \end{aligned} \right\} dx \quad (6)$$

The potential energy of elastic support is defined as:

$$U_s = \frac{1}{2}\sum k_s\left[w_0(x)\right]^2 \quad (7)$$

One dimensional homogeneous random field of elastic modulus is assumed:

$$E(x) = E_0\left[1 + r_E(x)\right] \quad (8)$$

where $r_E(x)$ is one dimensional homogeneous random field with zero mean. The autocorrelation function of the random field is:

$$R(\xi) = \sigma^2 \exp\left(-\frac{|\xi|}{d}\right) \quad (9)$$

where *σ, d,* are the coefficient of variation and the correlation distance for the random field of an elastic modulus, and the relative distance vector ξ is defined as $\xi = \mathbf{x}_j - \mathbf{x}_i$.

The randomness in the elastic modulus of the beam element is illustrated in Figure 2.

$$E_i = E_0(1 + r_i) \quad (10)$$

The random field of elastic modulus in the element is approximated by averaging random variables r_i at *n* points in the element:

$$E^e \approx E_0\left[1 + \frac{r_1 + r_2 + \ldots + r_n}{n}\right] \quad (11)$$

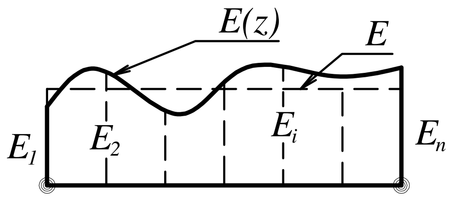

Fig. 2. Beam element with randomness in elastic modulus.

The stochastic stiffness matrix of the element is:

$$\begin{aligned}[K]^e &= \int_\Omega [B][D][B]d\Omega \\ &\approx \int_\Omega \left\{[B][D]_0\left[1+\frac{r_1+r_2+...+r_n}{n}\right][B]\right\}d\Omega\end{aligned} \tag{12}$$

The stiffness matrix of the structures consists of the stiffness matrix of the beam element and the stiffness of the elastic foundation:

$$[K]=[K]_{beam}+[K]_{support} \tag{13}$$

The free vibration equation is:

$$\left([K]-\omega^2[M]\right)\{X\}=\{0\} \tag{14}$$

The eigenvalue of the free vibration is $\lambda=\omega^2$. The stiffness matrix, the eigenvector, and the eigenvalue are expanded to Taylor series as:

$$\begin{aligned}[K]&=[K]_0+\sum_{i=1}^{Nr}\frac{\partial[K]}{\partial r_{ei}}r_{ei}+\frac{1}{2}\sum_{i=1}^{Nr}\sum_{j=1}^{Nr}\frac{\partial^2[K]}{\partial r_{ei}\partial r_{ej}}r_{ei}r_{ej}+\cdots \\ \lambda&=\lambda_0+\sum_{i=1}^{Nr}\frac{\partial\lambda}{\partial r_{ei}}r_{ei}+\frac{1}{2}\sum_{i=1}^{Nr}\sum_{j=1}^{Nr}\frac{\partial^2\lambda}{\partial r_{ei}\partial r_{ej}}r_{ei}r_{ej}+\cdots \\ \{X\}&=\{X\}_0+\sum_{i=1}^{Nr}\frac{\partial\{X\}}{\partial r_{ei}}r_{ei}+\frac{1}{2}\sum_{i=1}^{Nr}\sum_{j=1}^{Nr}\frac{\partial^2\{X\}}{\partial r_{ei}\partial r_{ej}}r_{ei}r_{ej}+\cdots\end{aligned} \tag{15}$$

and by substituting the series expressions in (6) into (5):

$$\begin{aligned}&\left(\begin{aligned}&[K]_0+\sum_{i=1}^{Nr}\frac{\partial[K]}{\partial r_{ei}}r_{ei}+\frac{1}{2}\sum_{i=1}^{Nr}\sum_{j=1}^{Nr}\frac{\partial^2[K]}{\partial r_{ei}\partial r_{ej}}r_{ei}r_{ej}+\cdots \\ &-\left\{\lambda_0+\sum_{i=1}^{Nr}\frac{\partial\lambda}{\partial r_{ei}}r_{ei}+\frac{1}{2}\sum_{i=1}^{Nr}\sum_{j=1}^{Nr}\frac{\partial^2\lambda}{\partial r_{ei}\partial r_{ej}}r_{ei}r_{ej}+\cdots\right\}[M]\end{aligned}\right)\times \\ &\left\{\{X\}_0+\sum_{i=1}^{Nr}\frac{\partial\{X\}}{\partial r_{ei}}r_{ei}+\frac{1}{2}\sum_{i=1}^{Nr}\sum_{j=1}^{Nr}\frac{\partial^2\{X\}}{\partial r_{ei}\partial r_{ej}}r_{ei}r_{ej}+\cdots\right\}=\{0\}\end{aligned} \tag{16}$$

Equation (16) is solved by comparing the terms of the random variables to obtain the solution:

$$\left[[K]_0 - \lambda_0[M]\right]X_0=\{0\} \tag{17}$$

and

$$\frac{\partial\lambda}{\partial r_{ei}}=\left(X_0\right)^T\frac{\partial[K]}{\partial r_{ei}}X_0 \tag{18}$$

The first-order approximation of the variance of the eigenvalues is:

$$\begin{aligned}&Var\left[\lambda\left(r_1,r_2,\cdots,r_{Nr}\right)\right] \\ &\cong E\left[\left(\lambda-E[\lambda]\right)\left(\lambda-E[\lambda]\right)\right] \\ &=\sum_{i=1}^{Nr}\sum_{j=1}^{Nr}\frac{\partial\lambda}{\partial r_{Ei}}\frac{\partial\lambda}{\partial r_{Ej}}R_E\left(\xi_{ij}\right)\end{aligned} \tag{19}$$

The variability of response COV of the eigenvalue is defined as:

$$COV=\frac{\sqrt{Var(\lambda)}}{E(\lambda)} \tag{20}$$

III. NUMERICAL EXAMPLES

In this section, a continuous beam on 5 elastic supports, as shown in Figure 1, is investigated. The geometrical and material parameters of the beam are: length L=10m, height h=0.6m, mean of Young's modulus $E=30\times10^5$MPa, coefficient of variation of a random field of elastic modulus σ=0.1, and mass density ρ=2400kg/m^3. To valid the proposed approach, it is compared with Monte Carlo simulation. The random field of stiffness of the elastic modulus is generated using the spectral representation method [36, 37]:

$$\begin{aligned}&r_E(x)=\sqrt{2}\sum_{n=0}^{N-1}A_n\cos(\omega_n x+\phi_n) \\ &A_n=\sqrt{2S_{ff}(\omega_n)\Delta\omega} \\ &\Delta\omega=\frac{\omega_u}{N} \\ &\omega_n=n\Delta\omega,\ n=0,1,2,...,N-1\end{aligned} \tag{21}$$

where ω_u denotes the upper cut-off frequency beyond with the power spectral density function $S_{ff}(\omega_n)$ of the random field of the stiffness of the elastic modulus.

The sample value of random field $r_E(x)$ is substituted to (11) and the stiffness matrix in (12) is deterministic. The free vibration equation (14) is repeatedly solved 10000 times to get the eigenvalue. Figure 3 shows the first and second mode shapes with samples of 3 cases of stiffness of elastic support $K_s=10^6,\ 10^7,\ 10^8$ N/m. It clearly shows that the first mode represents a symmetric mode shape and the second mode represents an antisymmetric mode shape. The mode shapes on Figure 3 clearly show the effect of the stiffness of support on mode shapes.

The effect of the correlation distance d of the random field on the variability of the eigenvalue is shown in Figure 4, where the results of the proposed formulation were compared with those of the Monte Carlo simulation using 10,000 samples with two cases of stiffness of elastic support, i.e. 10^7 and 10^8N/m. As seen in Figure 4, the COV varies depending on the correlation distance and the coefficient of stiffness of the elastic support. In all other cases, the obtained COV has a small value of the assumed standard deviation of the random field of elastic modulus. Also, the result on Figure 4 shows the good agreement between the proposed approach and Monte Carlo simulation.

(a)

(b)

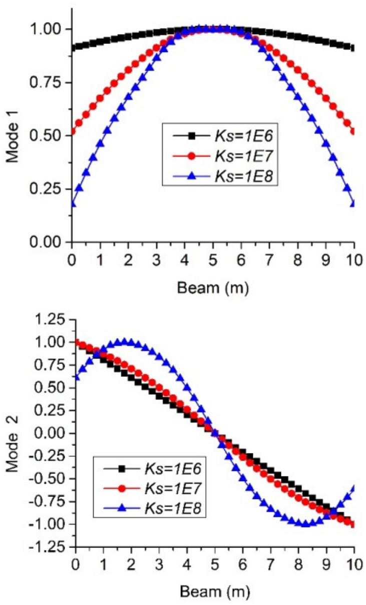

Fig. 3. Mode shapes. (a) First mode, (b) second mode.

(a)

(b)

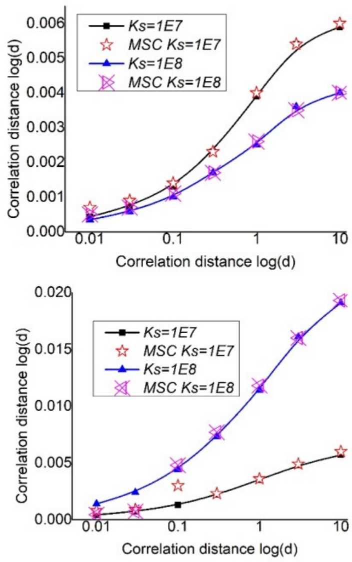

Fig. 4. Coefficient of variation of the eigenvalue. (a) First mode, (b) second mode.

Figure 5 illustrates the statistical probabilistic distribution of the eigenvalues of the first 2 modes obtained from Monte Carlo simulation corresponding with the stiffness of elastic support K_s =10^7N/m and two cases of correlation distance *d*=0.01 and *d*=0.1. On Figure 5, it is shown clearly that with smaller correlation distance, the histogram is quite scattered, whereas when the correlation distance is larger, the histogram is close to the normal distribution.

(a)

(b)

(c)

(d)

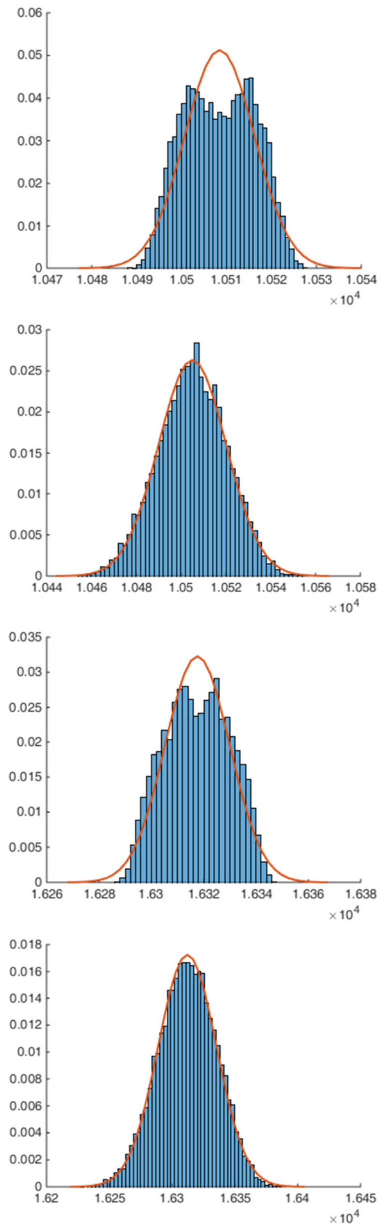

Fig. 5. Probabilistic distribution of the eigenvalue. (a) first mode with *d*=0.01, (b) first mode with *d*=0.1, (c) second mode with *d*=0.01, (d) second mode with *d*=0.1.

IV. CONCLUSIONS

In this paper, the stochastic finite element method was employed to investigate the response variability of the eigenvalue of the free vibration of a continuous beam on elastic support. The free vibration equation used higher order beam theory to establish finite element formulas. The first-order perturbation solution exhibits a good agreement with Monte Carlo simulation. The response variability of eigenvalues, given in terms of the coefficient of variation, is smaller of the input standard deviation of the random process of the elastic modulus of the beam.

ACKNOWLEDGMENT

This research is funded by University of Transport and Communications (UTC) under the grant number T2021-CT-007TĐ.

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