CUBO A Mathematical Journal
Vol. 6, № 4, (113-132). December 2004.

Search for geometric criteria for removable sets of bounded analytic functions

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ABSTRACT

This is a survey on results and in particular on recent development on the question: which compact subsets of the complex plane are removable for bounded analytic functions. We begin with classical results and here knowledge of only basic complex analysis is required. Later on some geometric measure theory and in particular very advanced theory of singular integrals enter into play. But since in this part we are not giving any detailed proofs, also a reader who is not familiar with these topics should be able to get an idea what is going on.

In the first parts of this paper I give several proofs for many rather easy and well-known results. I thus hope that a reader who is not familiar with the subject could gain some insight into it. In the later parts on more recent results some ideas of the proofs are only sketched. The complete details get often very complicated and are best studied in the original papers.

Recent lecture notes of Pajot [P1] cover this topic in much greater detail, also in historical perspective. There one can find many more references than we give here for the background and for further reading. Other recent surveys on this and related topics are [D4], [Me2], [P2], [V1] and [V2]. Much of the background material can be found in [G], [D2], [C1], [M2] and [Mu].

1 Removable sets

A classical theorem of Riemann says that an isolated singularity is removable for bounded complex analytic functions. This means that if

$$f : U(z_0, r) \setminus \{z_0\} \to \mathbf{C}$$

is analytic and bounded, then $f(z_0)$ can be defined so that f becomes analytic in the whole disc

$$U(z_0, r) = \{z \in \mathbf{C} : |z - z_0| < r\}.$$

There are many easy ways to see this, one is via the Cauchy integral formula. Let $0 < \varepsilon < \varrho < r$. Then for $\varepsilon < |z - z_0| < \varrho$,

$$f(z) = \frac{1}{2\pi i}\left[\int_{\partial U(z_0,\varrho)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\partial U(z_0,\varepsilon)} \frac{f(\zeta)}{\zeta - z}\, d\zeta\right].$$

Since f is bounded, the second integral tends to zero as $\varepsilon \to 0$. Hence for $0 < |z - z_0| < \varrho$,

$$f(z) = \frac{1}{2\pi i}\int_{\partial U(z_0,\varrho)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

The right hand side is analytic in $U(z_0, \varrho)$ and gives the required extension (since $\varrho < r$ was arbitrary, we get from this the extension to all of $U(z_0, r)$).

The main question we address in this survey is: with what kind of compact subsets of $\mathbf{C}$ can the singleton $\{z_0\}$ be replaced in the above result? We call such sets removable:

Definition 1.1 A compact set $E \subset \mathbf{C}$ is *removable* (for bounded analytic functions) if the following holds: if U is an open subset of $\mathbf{C}$ containing E and $f : U \setminus E \to \mathbf{C}$ is a bounded analytic function, then there is an analytic function $g : U \to \mathbf{C}$ such that $f(z) = g(z)$ for $z \in U \setminus E$.

Note that if E has interior points, then E is not removable. Indeed, if z_0 is an interior point of E, then $z \mapsto 1/(z - z_0)$ is bounded and analytic in $\mathbf{C} \setminus E$ but it does not have any analytic extension to $\mathbf{C}$.

The open sets U play no real role here. The following simple result means that it suffices to check the condition with $U = \mathbf{C}$:

Theorem 1.2 *A compact set $E \subset \mathbf{C}$ is removable if and only if every bounded analytic function $f : \mathbf{C} \setminus E \to \mathbf{C}$ is constant*

Proof. Suppose E is removable and $f : \mathbf{C} \setminus E \to \mathbf{C}$ is bounded and analytic. Since the interior of E is empty, the analytic extension of f to $\mathbf{C}$ is also bounded and by Liouville's theorem constant. Thus also f is constant.

Conversely, suppose that there are only constant bounded analytic functions in $\mathbf{C} \setminus E$. Let U be open with $E \subset U$ and $f : U \setminus E \to \mathbf{C}$ bounded and analytic. The point of the proof is that for any $z \in U \setminus E$, we can write

$$f(z) = \frac{1}{2\pi i}\left[\int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\Gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta\right],$$

where Γ_1 and Γ_2 are cycles consisting of finitely many smooth Jordan curves in U surrounding E and such that z is between them and in the outer domain of Γ_2. The integral along Γ_2 is independent of Γ_2 (as long as z lies in the outer domain of Γ_2), whence it defines a bounded analytic function in $\mathbf{C} \setminus E$. Hence it must be constant by our assumption. In the same way the integral along Γ_1 is independent of Γ_1, and gives an analytic function in U. From these facts we get the required extension of f.

The removable sets can also be characterized as the null-sets of the *analytic capacity* γ, which was introduced by Ahlfors in 1947 in [A]. For a compact set $E \subset \mathbf{C}$,

$$\gamma(E) = \sup_f \lim_{z\to\infty} |zf(z)|,$$

where the supremum is taken over all analytic functions $f : \mathbf{C} \setminus E \to \mathbf{C}$ such that $|f(z)| \leq 1$ for all $z \in \mathbf{C} \setminus E$ and $\lim_{z\to\infty} f(z) = 0$. Then E is removable if and only if $\gamma(E) = 0$. We leave the verification of this easy fact to the reader.

2 Painlevé's condition and Hausdorff measures

Now that we found other easy complex analytic reformulations of the removability, we return to the problem of finding geometric criteria. We started with Riemann's classical result that a singleton is a removable set. Of course, we get in the same way that any finite set is removable. Although the modification is trivial, let's do it. But it is slightly more convenient to do it using Theorem 1.2 than the definition. So let $E = \{z_1, \ldots, z_n\} \subset \mathbf{C}$ and let $f : \mathbf{C} \setminus E \to \mathbf{C}$ be bounded and analytic. We may assume $|f| \leq 1$ and $f(\infty) := \lim_{z\to\infty} f(z) = 0$. Pick $z \in \mathbf{C} \setminus E$, pick large radius R such that $E \subset U(0, R)$, and then pick for each j an $\varepsilon_j > 0$ such that $U(z_j, \varepsilon_j) \subset U(0, R) \setminus U(z, \varrho)$ where $\varrho = \operatorname{dist}(z, E)/2$. By the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\left[\int_{\partial U(0,R)} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \int_{\partial(\cup_{j=1}^n U(z_j,\varepsilon_j))} \frac{f(\zeta)}{\zeta - z}\, d\zeta\right].$$

Since $f(\infty) = 0$, the first integral tends to zero as $R \to \infty$. This gives

$$f(z) = -\frac{1}{2\pi i}\int_{\partial(\cup_{j=1}^n U(z_j,\varepsilon_j))} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \tag{2.1}$$

whence

$$|f(z)| \leq \frac{1}{2\pi}\sum_{j=1}^n \int_{\partial U(z_j,\varepsilon_j)} \frac{|f(\zeta)|}{|\zeta - z|}\, |d\zeta| \leq \frac{1}{2\pi}\sum_{j=1}^n \frac{1}{\varrho}\, 2\pi\varepsilon_j = \frac{1}{\varrho}\sum_{j=1}^n \varepsilon_j.$$

Since we can make $\sum_{j=1}^{n}\varepsilon_j$ as small as we please, we get $f(z)=0$. Thus f vanishes identically, and we have proved that E is removable.

Of course, in the above proof we could have chosen the same ε for each j. But we didn't in order to see that the above argument gives much more. The only thing we used about E is that we could cover it with discs $U(z_j,\varepsilon_j)$, $j=1,\dots,n$, so that $\sum_{j=1}^{n}\varepsilon_j$ is arbitrarily small. Clearly this is true for any compact countable set. But it is true also for many uncountable sets, for example, for any compact subset of $\mathbf{R}\subset\mathbf{C}$ with one-dimensional Lebesgue measure zero.

This covering condition was the sufficient condition Painlevé found for removability more than one hundred years ago. In more modern language it means that E has 1-dimensional Hausdorff measure zero. The integral dimensional Hausdorff measures $\mathcal{H}^k$, $k=1,2,\dots$, where defined by Carathéodory in 1914 to generalize the concept of length, area, and more generally k-dimensional area of k-dimensional surfaces. A few years later Hausdorff defined and studied s-dimensional Hausdorff measures $\mathcal{H}^s$ for $0\le s<\infty$. They are defined for $A\subset\mathbf{C}$ (or A in any metric space) by

$$\mathcal{H}^s(A)=\lim_{\delta\to 0}\inf\left\{\sum_{j=1}^{\infty}\operatorname{diam}(E_j)^s : A\subset\bigcup_{j=1}^{\infty}E_j,\ \operatorname{diam}(E_j)<\delta\right\}.$$

Then, for example, $\mathcal{H}^1(\Gamma)$ is the length of Γ if Γ is a rectifiable curve. For any s, $\mathcal{H}^s$ is an outer measure for which Borel sets are measurable. For this and other facts on Hausdorff measures the reader can consult for example [M2].

Now it is easy to check that for compact sets E Painlevé's condition means that the one-dimensional Hausdorff measure is zero: $\mathcal{H}^1(E)=0$ if and only if for any $\varepsilon>0$ there are discs $U(z_j,\varepsilon_j)$, $j=1,\dots,n$, such that

$$E\subset\bigcup_{j=1}^{n}U(z_j,\varepsilon_j)\qquad\text{and}\qquad\sum_{j=1}^{n}\varepsilon_j<\varepsilon.$$

Hence we can formulate Painlevé's result as

Theorem 2.1 *If $E\subset\mathbf{C}$ is compact with $\mathcal{H}^1(E)=0$, then E is removable.*

Let us now look for some non-removable sets. We already observed that all compact sets with interior points are such. So are all compact sets with positive area, i.e., with positive two-dimensional Lebesgue measure $\mathcal{L}^2(E)$. To see this, it is enough to check that the function f,

$$f(z)=\int_E\frac{1}{\zeta-z}\,d\mathcal{L}^2\zeta,\qquad z\in\mathbf{C}\setminus E,$$

is bounded, analytic and not constant in $\mathbf{C}\setminus E$. Such a function is called the Cauchy transform of the measure $\mathcal{L}^2|E$. Cauchy transforms are very important in the study of removability. For a general finite Borel measure σ, which could be non-negative, real, or complex, the Cauchy transform C_σ of σ is defined by

$$C_\sigma(z)=\int\frac{1}{\zeta-z}\,d\sigma\zeta,$$

at the points z where the integral exists. We often study measures σ which live on some compact set E, that is, $\sigma(A) = 0$ for all Borel sets $A \subset \mathbf{C} \setminus E$. Then C_σ is defined and analytic in $\mathbf{C} \setminus E$. It is constant only if $\sigma \equiv 0$. But it is not always bounded.

Let us study the possible boundedness of C_μ for some non-negative finite measures μ. We already noticed that C_μ is bounded when $\mu = \mathcal{L}^2 | E$ for some compact set E. One easy way to check this is to use the obvious inequality

$$\mathcal{L}^2\big(E \cap U(z,r)\big) \leq \pi r^2 \qquad \text{for all } z \in \mathbf{C},\ r > 0.$$

But less is enough: if for some $s > 1$, and $0 < c < \infty$,

$$\mu\big(U(z,r)\big) \leq c r^s \qquad \text{for } z \in \mathbf{C} \text{ and } r > 0,$$

then $\mathbf{C}_\mu$ is bounded. In fact, a well-known formula for non-negative functions g reads as,

$$\int g\, d\mu = \int_0^\infty \mu\big(\{x : g(x) > t\}\big)\, dt.$$

It is an easy consequence of Fubini's theorem. Thus

$$\begin{aligned} |C_\mu(z)| &= \left| \int \tfrac{1}{\zeta - z}\, d\mu\zeta \right| \leq \int \tfrac{1}{|\zeta - z|}\, d\mu\zeta \\ &= \int_0^\infty \mu\Big(\Big\{\zeta : \tfrac{1}{|\zeta - z|} > t\Big\}\Big)\, dt = \int_0^\infty \mu\big(U(z, 1/t)\big)\, dt \\ &\leq \int_0^1 u^{-2} \mu\big(U(z,u)\big)\, du + \int_1^\infty u^{-2} \mu(\mathbf{C})\, du \\ &\leq c \int_0^1 u^{s-2}\, du + \mu(\mathbf{C}) = \tfrac{c}{s-1} + \mu(\mathbf{C}). \end{aligned}$$

To study the size of general sets, one can use the whole scale of Hausdorff measures $\mathcal{H}^s$, $0 \leq s \leq 2$, ($\mathcal{H}^s(\mathbf{C}) = 0$ if $s > 2$). One easily checks that

$$t < s, \quad \mathcal{H}^t(A) < \infty \quad \text{implies } \mathcal{H}^s(A) = 0.$$

This leads to the definition of the Hausdorff dimension $\dim A$;

$$\begin{aligned} \dim A &= \inf\{s : \mathcal{H}^s(A) = 0\} \\ &= \sup\{t : \mathcal{H}^t(A) = \infty\}. \end{aligned}$$

There is a classical result of Frostman from the 1930's which says that for Borel sets A, $\mathcal{H}^s(A) > 0$ if and only if there is a non-negative Borel measure μ on $A \subset \mathbf{C}$ such that $\mu(A) > 0$ and $\mu\big(B(x,r)\big) \leq r^s$ for all $x \in \mathbf{C}$, $r > 0$, see [M2, Theorem 8.8].

Suppose that $E \subset \mathbf{C}$ is compact with $\dim E > 1$. Then $\mathcal{H}^s(E) > 0$ for some $s > 1$ and by Frostman's result there is μ on E for which $\mu\big(B(z,r)\big) \leq r^s$ for $z \in \mathbf{C}$, $r > 0$. The estimates we had a little earlier show then that C_μ is bounded in $\mathbf{C} \setminus E$. Thus we have proved

Theorem 2.2 *If $E \subset \mathbf{C}$ is compact and $\dim E > 1$, then E is not removable.*

To get some examples of sets to which this theorem applies, we consider Cantor sets C_s, $0 < s < 2$, on the unit square $Q_0 = [0,1] \times [0,1]$. They are constructed as follows. Let $\lambda \in (0, 1/2)$ be defined by

$$s = \log 4 / \log(1/\lambda).$$

Let $Q_{1,i} \subset Q_0$, $i = 1, \dots, 4$, be the four closed squares in the corners of Q_0 with side-length λ. Then inductively if the squares $Q_{k,i}$, $i = 1, \dots, 4^k$, of side-length λ^k have been constructed, each of them gives birth to 4 new squares of side-length λ^{k+1}, and altogether we get 4^{k+1} squares $Q_{k+1,i}$, $i = 1, \dots, 4^{k+1}$. Then C_s is defined by

$$C_s = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{4^k} Q_{k,i}.$$

It is not difficult to see that $0 < \mathcal{H}^s(C_s) < \infty$. Moreover, C_s is an example of a so-called AD-regular set: there is a constant C, $1 < C < \infty$, such that

$$r^s/C \le \mathcal{H}^s\big(C_s \cap B(x,r)\big) \le C r^s \quad \text{for } x \in C_s,\ 0 < r < 1.$$

In particular, when $s > 1$ (i.e., $\lambda > 1/4$), $\dim C_s > 1$. Hence by Theorem 2.2, E is not removable. For $s < 1$ (and $\lambda < 1/4$), $\dim C_s < 1$, and so $\mathcal{H}^1(C_s) = 0$. Thus by Theorem 2.1, C_s is removable. Let us now look at the most interesting case $s = 1$ (and $\lambda = 1/4$). Then C_1 has positive and finite "length" (one-dimensional Hausdorff measure), but it is nothing like a rectifiable curve. It is a standard example of a purely unrectifiable set:

$$\mathcal{H}^1(C_1 \cap \Gamma) = 0$$

for every rectifiable curve Γ.

We can use C_1 to show that the converse of Theorem 2.1 is false: C_1 is removable although $\mathcal{H}^1(C_1) > 0$. To see why this is so, let us first look more generally at compact sets with finite $\mathcal{H}^1$ measure.

So let $E \subset \mathbf{C}$ be compact with $\mathcal{H}^1(E) < \infty$, and let $f : \mathbf{C} \setminus E \to \mathbf{C}$ be an analytic function such that $|f(z)| \le 1$ for $z \in \mathbf{C} \setminus E$ and $f(\infty) = 0$. As in the discussion of Painlevé's result, we get a representation as in (2.1). Using the fact that $\mathcal{H}^1(E) < \infty$, we can for each $m = 1, 2, \dots$ choose as the discs $U(z_j, \varepsilon_j)$ discs $U_{m,j}$, $j = 1, \dots, k_m$, such that

$$\operatorname{diam}(U_{m,j}) < 1/m \qquad \text{and} \qquad \sum_{j=1}^{k_m} \operatorname{diam}(U_{m,j}) < \mathcal{H}^1(E) + 1.$$

Then the formula (2.1) can be re-written as

$$f(z) = \int \frac{1}{\zeta - z}\, d\sigma_m \zeta, \tag{2.2}$$

where σ_m is the complex measure for which

$$\int g\, d\sigma_m = -\frac{1}{2\pi i} \int_{\partial(\cup_{j=1}^{k_m} U_{m,j})} g(\zeta)\, f(\zeta)\, d\zeta$$

for all continuous functions g. The condition $\sum_{j=1}^{k_m} \operatorname{diam}(U_{m,j}) < \mathcal{H}^1(E)+1$ guarantees that the total variations of the measures σ_m are uniformly bounded, whence we can extract a weakly converging subsequence $\sigma_{m_j} \to \sigma$. Then

$$f(z) = \int \frac{1}{\zeta - z}\, d\sigma\zeta, \qquad z \in \mathbf{C} \setminus E.$$

A closer, but not difficult, investigation shows (see, e.g., [M2, Ch. 19]) that σ is absolutely continuous with respect to $\mathcal{H}^1$ with bounded Radon–Nikodym derivative. This means that there is a bounded Borel function $\varphi : E \to \mathbf{C}$ such that

$$f(z) = \int_E \frac{\varphi(\zeta)}{\zeta - z}\, d\mathcal{H}^1\zeta. \tag{2.3}$$

The above representation formula is very useful in the study of the removability of sets with finite $\mathcal{H}^1$ measure. After that we can often forget about the analytic functions and use only the boundedness of the right hand side of (2.3). In fact, we have more than this. By some simple estimates, cf. [M2, Lemma 19.14], one sees that the corresponding maximal function is bounded, or in other words, there is a constant C, $1 < C < \infty$, such that

$$C_\mu^*\varphi(z) := \sup_{\varepsilon>0} \left| \int_{\mathbf{C}\setminus B(z,\varepsilon)} \frac{\varphi(\zeta)}{\zeta - z}\, d\mu\zeta \right| < C \qquad \text{for } z \in \mathbf{C}, \tag{2.4}$$

where $\mu = \mathcal{H}^1|E$.

Let us now see how we can use this for the Cantor set C_1. To show that C_1 is removable, we need to show that if $f : \mathbf{C} \setminus C_1 \to C$ is analytic, $|f| \leq 1$ and $f(\infty) = 0$, then $f \equiv 0$. For such an f we find φ as above, and we have to show that $\varphi = 0$ $\mathcal{H}^1$ almost everywhere on C_1. Suppose not, and first suppose that even $\varphi \equiv 1$ on C_1. Then this is very easy. If we take $z = 0$,

$$\operatorname{Re} \frac{1}{\zeta - z} = \operatorname{Re} \frac{\bar{\zeta}}{|\zeta|^2} \geq 0 \qquad \text{for all } \zeta \in C_1.$$

Moreover, $\operatorname{Re} 1/\zeta \geq 4^{k-2}$ in some square Q_{k,i_k} in each generation (in the one which touches the real axis and is the second closest to 0). Hence for every $m = 2, 3, \ldots,$

$$\operatorname{Re} \int_{C_1 \setminus U(0,4^{-m-1})} \frac{1}{\zeta}\, d\mathcal{H}^1\zeta \geq \sum_{k=1}^{m} 4^{k-2}\, \mathcal{H}^1(C_1 \cap Q_{k,i_k}) = \sum_{k=1}^{m} 4^{k-2}\, 4^{-k} = m/16,$$

which violates (2.4). Of course, the assumption $\varphi \equiv 1$ is a huge oversimplification. But vaguely speaking the general case can be reduced to this. First, any measurable function is approximately continuous almost everywhere. Hence near any typical point of C_1, φ looks like a constant except for a set of small measure. But 0, which we used above, is a very special point of C_1. On the other hand, typical points of C_1 look like corner points (such as 0) at some arbitrary small scales. A precise statement of this sort says that given any N, then $\mathcal{H}^1$ almost every point $z \in C_1$ belongs infinitely often N consequtive times to the left lower hand corner square in the construction of C_1. This is easy to prove. Using these principles it is possible to give a rigorous proof for the fact that C_1 is removable.

The above arguments on the self-similar Cantor set C_1 can be generalized. We essentially used two facts. First, that C_1 is AD-regular. Recall that this meant that there is C, $1 \leq C < \infty$, such that

$$r/C \leq \mathcal{H}^1\big(E \cap B(z,r)\big) \leq Cr \qquad \text{for } z \in E,\ 0 < r < \operatorname{diam}(E). \tag{2.5}$$

Secondly, we used the fact that near its typical points, C_1 almost lies in a half-plane at arbitrarily small scales and also a considerable part of it is not too close to the boundary of that half-plane. For a general E we can still find many places where locally E almost lies in a half-plane by touching with discs from the complement. That is, we use discs $U(a,r)$ such that $U(a,r) \cap E = \emptyset$ but $\partial U(a,r) \cap E \neq \emptyset$. If we make an additional assumption that at small scales E is not well approximated by lines, we can extend the argument we sketched above for C_1. A more general result can be obtained using a result of Besicovitch, see [M2, Theorem 19.17] and the references there:

Theorem 2.3 *Let $E \subset \mathbf{C}$ be compact with $\mathcal{H}^1(E) < \infty$. Suppose that for $\mathcal{H}^1$ almost all $a \in E$,*

$$\liminf_{r \to 0} \mathcal{H}^1\big(E \cap U(a,r)\big)/r > 0 \tag{2.6}$$

and there are positive numbers r_0, δ and ε (depending on a) such that

$$\mathcal{H}^1\big(E \cap U(a,r) \cap \{z : \operatorname{dist}(z,L) > \delta r\}\big) > \varepsilon r$$

for all lines L through a and for all $0 < r < r_0$. Then E is removable.

Note that according to one of the basic properties of Hausdorff measures we have for any E with $\mathcal{H}^1(E) < \infty$,

$$1/2 \leq \limsup_{r \to 0} \mathcal{H}^1\big(E \cap U(a,r)\big)/r \leq 1$$

for $\mathcal{H}^1$ almost all $a \in E$, cf. [M2, Theorem 6.2]. So the lower density assumption (2.6) means that E is AD-regular in a non-quantitative sense.

3 Subsets of rectifiable curves

Theorem 2.3 is about as far as we can get in finding sufficient criteria for removability without new effective tools. We come back to these later but let us first search for more non-removable sets. We already know that all compact sets with Hausdorff dimension bigger than one are such. All non-constant rectifiable curves (that is, curves Γ with $0 < \mathcal{H}^1(\Gamma) < \infty$) are examples of non-removable sets with positive and finite $\mathcal{H}^1$ measure. In fact, for any compact connected set $K \subset \mathbf{C}$, which is not a singleton, the Riemann mapping theorem gives a bounded non-constant analytic function in $\mathbf{C} \setminus K$.

But let us construct such a function more concretely with the Cauchy integral in the simple case of line segments, say $[0,1] \subset \mathbf{R} \subset \mathbf{C}$. Take any continuously differentiable function $\varphi : [0,1] \to \mathbf{R}$ such that $\varphi(0) = \varphi(1) = 0$. Then

$$z \mapsto \int_0^1 \frac{\varphi(t)}{t-z}\, dt \tag{3.1}$$

is a bounded and analytic function $\mathbf{C} \setminus [0,1]$. This is an exercise, and not very difficult. It takes more work to show that if $E \subset \mathbf{R}$ is compact with $\mathcal{H}^1(E) > 0$ (for $E \subset \mathbf{R}$, $\mathcal{H}^1(E)$ is just the Lebesgue measure of E), then such a φ can still be constructed. An easier way avoiding the construction is to check first that the imaginary part of g,

$$g(z) = \int_E \frac{1}{t-z}\, d\mathcal{H}^1 t$$

is bounded; this is easy. Thus g maps to a strip and this strip can be mapped conformally onto the unit disc.

These arguments show that any compact subset E of a line with $\mathcal{H}^1(E) > 0$ is non-removable. It is hardly surprising that lines can be replaced by sufficiently smooth curves. But how smooth? Concrete but more laboursome arguments with Cauchy integrals work up to C^2; twice continuously differentiable curves, and a bit further to $C^{1+\varepsilon}$, namely to the case where the tangent of that curve varies in a Hölder continuous manner, see, e.g. [C1, p. 100]. But C^1 is already as big a problem as general rectifiable curves which always have Lipschitz parametrizations. Still the result is true for them, see [C1, Chapter VII].

Theorem 3.1 *Let $\Gamma \subset \mathbf{C}$ be a rectifiable curve and $E \subset \Gamma$ a compact set with $\mathcal{H}^1(E) > 0$. Then E is not removable for bounded analytic functions.*

This means that there is a non-constant bounded analytic function $f : \mathbf{C} \setminus E \to \mathbf{C}$. But nobody knows how to construct it. All the proofs rely in one way or another on duality arguments involving the Hahn–Banach theorem. Another crucial tool is the Cauchy singular integral operator.

The statement of Theorem 3.1 has been called the Denjoy conjecture. In fact, Denjoy claimed long ago to have proven it but there was a gap in his proof. The final proof followed when Calderón proved in 1977 in [C] that the Cauchy singular integral operator on Lipschitz graphs with small Lipschitz constant is bounded in L^2. This together with earlier works of several mathematicians gave Theorem 3.1. What does Calderón's theorem mean and what is the Cauchy singular integral operator on subsets of $\mathbf{C}$?

4 The Cauchy singular integral operator

Let us look at this question more generally. Let μ be a locally finite non-negative Borel measure in $\mathbf{C}$, that is, compact sets have finite μ measure and Borel sets are μ

measurable. Formally, the Cauchy singular integral operator C_μ related to μ maps a function g to

$$C_\mu g(z) = \int \frac{g(\zeta)}{\zeta - z}\, d\mu\zeta.$$

If z is outside the support of μ this is just the Cauchy transform of $gd\mu$, and it is well defined at least if $\int |g|\, d\mu < \infty$. But now we want to study it also at the points of the support of μ, and our main interest is in the case where μ is somehow one-dimensional; for example, length measure on a rectifiable curve or more generally one-dimensional Hausdorff measure $\mathcal{H}^1$ on a set E with $0 < \mathcal{H}^1(E) < \infty$. Then it is not clear at all that the above integral should exist, and often it does not, even for very smooth functions g or even for constant functions. But if μ is the length measure on a sufficiently smooth curve and g is sufficiently smooth, then $C_\mu g(z)$ exists in the sense of principal values:

$$C_\mu g(z) = \lim_{\varepsilon \to 0} \int_{\mathbf{C} \setminus B(z,\varepsilon)} \frac{g(\zeta)}{\zeta - z}\, d\mu\zeta.$$

For example, if μ is the Lebesgue measure on $\mathbf{R}$ and g is Hölder continuous with compact support, the above limit exists (and it is π times the Hilbert transform of g). This is due to the cancellation which allows us to write

$$\int_{\varepsilon < |\zeta - z| < 1} \frac{g(\zeta)}{\zeta - z}\, d\zeta = \int_{\varepsilon < |\zeta - z| < 1} \frac{g(\zeta) - g(z)}{\zeta - z}\, d\zeta.$$

Note that we are now in a very similar situation as that where we discussed the non-removability of subsets of smooth curves with positive length. Indeed, the boundedness of the function in (3.1) is due to similar cancellations.

If μ is the length measure on a sufficiently smooth curve ($C^{1+\varepsilon}$ is again enough) so that $C_\mu g(z)$ exists for all, say, C^∞ functions, then the L^2-boundedness of C_μ means (and is true) that it is bounded in such a subspace of smooth functions. It can then be extended as a bounded operator to all of $L^2(\mu)$. But the L^2-boundedness among C^∞-functions is not trivial even for the Hilbert transform, i.e., when μ is the length measure on $\mathbf{R}$. However, with the help of Fourier transform it is not too hard either for the Hilbert transform.

If μ is the length measure on a general rectifiable curve, the existence of $C_\mu g(z)$ is not at all clear for smooth functions, and it need not even be true at every point of the curve. However, $C_\mu g(z)$ exists in this case almost everywhere on the curve for all $g \in L^1(\mu)$. This is best to prove as a consequence of the L^2-boundedness and the general Calderón–Zygmund theory of singular integrals. But how to define the L^2-boundedness.

There are various ways and we take the one which can be used for any finite Borel measure μ:

Definition 4.1 Let μ be a finite Borel measure on $\mathbf{C}$. We say that the Cauchy singular integral operator C_μ is bounded in $L^2(\mu)$ if all the truncated operators $C_{\mu,\varepsilon}$;

$$C_{\mu,\varepsilon}\, g(z) = \int_{\mathbf{C} \setminus B(z,\varepsilon)} \frac{g(\zeta)}{\zeta - z}\, d\mu\zeta,$$

are uniformly bounded in $L^2(\mu)$. In other words, there is $C < \infty$ such that

$$\int \left| \int_{\mathbf{C}\setminus B(z,\varepsilon)} \frac{g(\zeta)}{\zeta - z}\, d\mu\zeta \right|^2 d\mu z \le C \int |g|^2\, d\mu$$

for all $g \in L^2(\mu)$ and all $\varepsilon > 0$.

Then Calderón's theorem says that C_μ is bounded in $L^2(\mu)$ if μ is the length measure on a Lipschitz graph $\{x + i\,\varphi(x) : x \in I\}$ where $\varphi : I \to \mathbf{R}$ has small Lipschitz constant. As said above, this was the final step needed to finish the proof of the Denjoy conjecture (Theorem 3.1). Let us try to say a few words how this goes.

So we take a compact set E in a rectifiable curve Γ with $\mathcal{H}^1(E) > 0$. Using some basic properties of rectifiable curves we can first show that some compact subset of E with positive length lies on a Lipschitz graph with small Lipschitz constant. Thus we can assume that Γ is such to begin with, and we can use the L^2-boundedness. The general Calderón–Zygmund theory of singular integrals then applies and tells us that the L^2-boundedness has many consequences. For example, the corresponding maximal operator C_μ^*, see (2.4) is bounded in any L^p for any $1 < p < \infty$. It need not be bounded in L^1, but there is a substitute, the weak type inequality:

$$\mathcal{H}^1\big(\{z \in \Gamma : |C_\mu^* g(z)| > \lambda\}\big) \le \frac{C}{\lambda} \int |g|\, d\mu$$

for $g \in L^1(\Gamma)$ and $\lambda > 0$. It is here where we can bring in the Hahn–Banach theorem to produce $\varphi \in L^\infty(\mu)$ such that $C_\mu^* \varphi \in L^\infty(\mu)$. Then the Cauchy transform of $\varphi\, d\mu$ is the desired bounded analytic function in $\mathbf{C} \setminus \Gamma$. This is pretty vague, and one really applies Hahn–Banach theorem to some regularized Cauchy operators. See [C1, Chapter VII] for more details.

Calderón's theorem was extended to all Lipschitz graphs by Coifman, McIntosh and Meyer in 1982. Finally David gave in [D1] characterization of curves for which this holds:

Theorem 4.2 *Let $\Gamma \subset \mathbf{C}$ be a rectifiable curve and $\mu = \mathcal{H}^1|\Gamma$ the length measure on Γ. Then C_μ is bounded in $L^2(\mu)$ if and only if Γ is AD-regular.*

Note that for curves Γ the AD-regularity means simply that there is $C < \infty$ such that $\mathcal{H}^1\big(\Gamma \cap B(z,r)\big) \le Cr$ for $z \in \mathbf{C}$, $r > 0$; the lower bound is automatic for curves.

Definition of the L^2-boundedness above was given for general measures μ, so we can ask for generalizations of the above theorem. For example, what kind of sets with finite $\mathcal{H}^1$ measure can be used to replace Γ. Among the class of AD-regular (recall (2.5)) sets a complete answer has been given by Melnikov, Verdera and myself in [MMV]:

Theorem 4.3 *Let $E \subset \mathbf{C}$ be a closed AD-regular set and $\mu = \mathcal{H}^1|E$. Then C_μ is bounded in $L^2(\mu)$ if and only if E is contained in an AD-regular curve.*

Such sets E are called uniformly rectifiable. Their theory, and in particular the theory of their m-dimensional versions in $\mathbf{R}^n$, has been developed by David and Semmes, see [DS].

5 Menger curvature and Cauchy integral

We shall see soon that Theorem 4.3 can be used to obtain new information about removable sets. Let us now discuss the proof of Theorem 4.3. One direction is clear: if E is contained in an AD-regular curve, it follows immediately from David's theorem (Theorem 4.2) that C_μ is bounded in $L^2(\mu)$. For the other direction we need a new tool which has recently given much more, as we shall see. This is the so-called Menger curvature. It is defined for triples of points in $\mathbf{C}$. So let $x, y, z \in \mathbf{C}$. If they do not lie on the same line, there is a unique circle passing through them. Let R be the radius of this circle. The Menger curvature of the triple (x, y, z) is then defined as

$$c(x,y,z) = \frac{1}{R}.$$

If x, y and z lie on the same line, we set

$$c(x,y,z) = 0.$$

In the mid 1990's Melnikov found the surprising formula in [Me1] which relates this to the Cauchy kernel: for $z_1, z_2, z_3 \in \mathbf{C}$,

$$c(z_1,z_2,z_3)^2 = \sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}}. \tag{5.1}$$

Here the sum is over all six permutations of $\{1, 2, 3\}$.

The proof of this formula is an exercise. It can be split into two exercises; one in complex numbers and one in elementary geometry. The first exercise is to show that the above sum over σ's equals

$$\left(\frac{4\,\mathrm{area}\,\big(T(z_1,z_2,z_3)\big)}{|z_1 - z_2|\,|z_1 - z_3|\,|z_2 - z_3|}\right)^2 \tag{5.2}$$

where $T(z_1, z_2, z_3)$ is the triangle whose vertices are z_1, z_2, z_3. This is not as hard as it may look. By some symmetries one can quickly pair some terms of the sum leading to a sum of three terms. Further, one can easily reduce to the case where $z_1 = 0$, $z_2 = 1$ and z_3 is a general complex number.

The second exercise says that (5.2) equals $1/R^2$, where R is the above radius. This is not very difficult. It is actually a theorem of the ancient Greeks.

Melnikov and Verdera used in [MV] the identity (5.1) to give a new proof of the L^2-boundedness of the Cauchy operator on Lipschitz graphs. At that time there already were 10 or so proofs (see in particular [Mu]), but this is one of the nicest. The trick is to integrate with respect to all three variables.

We now do this integration to continue the explanation of the proof of Theorem

4.3. Let μ be a finite Borel measure on $\mathbf{C}$. Formally, by Fubini's theorem and (5.1),

$$\begin{aligned}\int\left|\int \tfrac{1}{\zeta-z}\,d\mu\zeta\right|^2 d\mu z &= \iint \tfrac{1}{z_1-z_3}\,d\mu z_1\overline{\int \tfrac{1}{z_2-z_3}\,d\mu z_2}\,d\mu z_3\\ &= \iiint \tfrac{1}{(z_1-z_3)\overline{(z_2-z_3)}}\,d\mu z_1\,d\mu z_2\,d\mu z_3\\ &= \tfrac{1}{6}\iiint \textstyle\sum_\sigma \tfrac{1}{(z_{\sigma(1)}-z_{\sigma(3)})\overline{(z_{\sigma(2)}-z_{\sigma(3)})}}\,d\mu z_1\,d\mu z_2\,d\mu z_3\\ &= \tfrac{1}{6}\iiint c(z_1,z_2,z_3)^2\,d\mu z_1\,d\mu z_2\,d\mu z_3.\end{aligned}$$

This was formal because the first integrals don't really exist and we should replace some of them by truncated integrals as in the definition of the L^2-boundedness. This is a technicality which brings in an error term that can easily be controlled. The main point is that we started from the L^2-integral which is finite by our assumption. Hence also the curvature integral is finite. Using the AD-regularity we can conclude from this that $c(z_1, z_2, z_3)$ cannot be big too often. But when z_1, z_2 and z_3 are close to each other, this means that they must be nearly collinear and we are in a position where we can try to start to build the required AD-regular curve. This is not quite easy, but such constructions were made earlier in closely related situations by Jones in [J] and by David and Semmes for example in [DS].

6 Rectifiable and purely unrectifiable sets

Before explaining how Theorem 4.3 can be used to get more information about removable sets for bounded analytic functions, let us have a quick overview about the kind of geometric properties of sets we shall be looking at. We consider $\mathcal{H}^1$ measurable subsets E of the plane with $\mathcal{H}^1(E) < \infty$. We have already looked at some very different ones; rectifiable curves and the Cantor set C_1. All such sets E can be classified into two classes where, in a rather rough sense, properties of the sets in the first class are similar to those of rectifiable curves, and in the second to those of C_1.

Definition 6.1 Let $E \subset \mathbf{C}$ with $\mathcal{H}^1(E) < \infty$. We say that E is *rectifiable* if there are rectifiable curves $\Gamma_1, \Gamma_2, \dots$ such that

$$\mathcal{H}^1\left(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\right) = 0.$$

We say that E is *purely unrectifiable* if

$$\mathcal{H}^1(E \cap \Gamma) = 0$$

for every rectifiable curve Γ.

It is easy to show that any $\mathcal{H}^1$ measurable set E with $\mathcal{H}^1(E) < \infty$ can be written as $E = R \cup P$ where R is rectifiable and P is purely unrectifiable. It is also true, but not always so easy, that rectifiable sets behave in many ways as rectifiable curves and purely unrectifiable sets behave in completely opposite ways. For example, if the concept of tangent is defined in an approriate approximate sense, then E is rectifiable if and only if it has tangent at almost all of its points, and E is purely unrectifiable if and only if it fails to have tangent at almost all of its points. Another characterization can be given in terms of orthogonal projections. Let $\theta \in S^1 = \{z \in \mathbf{C} : |z| = 1\}$ be a unit vector and p_θ the orthogonal projection onto the line $L_\theta = \{t\theta : t \in \mathbf{R}\}$:

$$p_\theta(z) = (\theta \cdot z)\,\theta.$$

Then (assuming all the time that E is $\mathcal{H}^1$ measurable and $\mathcal{H}^1(E) < \infty$) E is purely unrectifiable if and only if

$$\mathcal{H}^1\big(p_\theta(E)\big) = 0 \qquad \text{for } \mathcal{H}^1 \text{ almost all } \theta \in S^1.$$

These results, and many more, were proven by Besicovitch in the 1920's and 30's. For this theory see [F], [Fe] or [M2]. This can be considered as the beginning of geometric measure theory. Besicovitch used the terminology of regular and irregular sets. We have here adopted (essentially) Federer's terminology, who generalized most of Besicovitch's theory to m-dimensional sets in $\mathbf{R}^n$.

Let us now see how much at this point we can say about removability and non-removability in terms of rectifiability. Suppose a compact set E is not purely unrectifiable. Then $\mathcal{H}^1(E \cap \Gamma) > 0$ for some rectifiable curve Γ. Hence $E \cap \Gamma$ is not removable by Theorem 3.1, and consequently neither is E. Thus we have:

Let $E \subset \mathbf{C}$ be compact with $\mathcal{H}^1(E) < \infty$. If E is removable, it is purely unrectifiable.

What about the converse statement: are all purely unrectifiable compact sets with finite $\mathcal{H}^1$ measure removable? We already know that this is true for the Cantor set and more generally in the case of Theorem 2.3. A remarkable fact proven by David in [D3] is that it is true generally:

Theorem 6.2 *Let $E \subset \mathbf{C}$ be compact with $\mathcal{H}^1(E) < \infty$. Then E is removable if and only if it is purely unrectifiable.*

Let us first discuss the proof in the case of AD-regular sets. Then this is a consequence of Theorem 4.3 and some general results and methods in the Calderón–Zygmund theory of singular integrals. We should show that if E is not removable, then it is not purely unrectifiable. We start with a non-constant bounded analytic function in $\mathbf{C} \setminus E$. As in Section 2 this leads to a function $\varphi \in L^\infty(E)$ such that $\int_E \varphi(\zeta)/(\zeta - z)\, d\mathcal{H}^1\zeta$ is bounded in $\mathbf{C} \setminus E$. This is essentially the same as to say that $C_\mu \varphi \in L^\infty(\mu)$, where $\mu = \mathcal{H}^1|E$. In order to apply Theorem 4.3 we need the L^2-boundedness of C_μ. We are now rather close to that because of a general $T(b)$-theorem of David, Journé and Semmes, see [D2]. It applies to very general singular integral operators T and says that if T maps some bounded function b with $\operatorname{Re} b \geq \delta > 0$ (in fact, somewhat less is needed) to L^∞ (or even BMO), then T is bounded in L^2. We have now that C_μ maps φ to L^∞, but we have no information on $\operatorname{Re}\varphi$. However, Christ showed in [C2] how it is possible to modify E to another AD-regular set F and φ to a bounded function ψ on F such that $\mathcal{H}^1(E \cap F) > 0$, $\operatorname{Re}\psi \geq \delta > 0$ and

$C_\nu \psi \in \mathrm{BMO}$ with $\nu = \mathcal{H}^1|F$. Thus C_ν is bounded in $L^2(\nu)$, and by Theorem 4.3, F, and hence also $E \cap F$, is contained in a rectifiable curve so that E cannot be purely unrectifiable.

The proof of Theorem 6.2 follows similar lines but there are a lot of technical complications. We still have that $C_\mu \varphi \in L^\infty(\mu)$ for some $\varphi \in L^\infty(\mu)$ as above. The modification from E and φ to F and ψ can be done, but it is now much more complicated, see [DM]. But even after that the $T(b)$-theorems that were available before the late 1990's could not be applied since they required that the underlying measure is doubling;

$$\mu\big(B(x,2r)\big) \leq C\,\mu\big(B(x,r)\big)$$

for all x in the support of μ and for all $r > 0$. The final step of the proof of Theorem 6.2 was David's proof in [D3] of a $T(b)$-theorem in the non-doubling case. After that we know that C_μ is bounded in $L^2(\mu)$ with $\mu = \mathcal{H}^1|E \cap F$. Then the argument in Section 5 yields that $\int c^2\, d\mu^3 < \infty$. Another hard problem was to show that this implies rectifiability. This was done a little earlier in [L]:

Theorem 6.3 *Let $E \subset \mathbf{C}$ be $\mathcal{H}^1$ measurable with $\mathcal{H}^1(E) < \infty$. If*

$$\iiint c(x,y,z)^2\, d\mu x\, d\mu y\, d\mu z < \infty,$$

then E is rectifiable.

All these together give the proof of Theorem 6.2. Nazarov, Treil and Volberg developed a different powerful method for the analytic parts of this proof in [NTV].

Coming back to Theorem 6.2 and the properties of purely unrectifiable sets, the result means that when $\mathcal{H}^1(E) < \infty$, E is removable if and only if

$$\mathcal{H}^1\big(p_\theta(E)\big) = 0 \qquad \text{for } \mathcal{H}^1 \text{ almost all } \theta \in S^1. \tag{6.4}$$

Vitushkin conjectured this already in [Vi] in the 1950's. And even without the condition $\mathcal{H}^1(E) < \infty$, but then this fails. I showed in the 1980's in [M1] that (6.4) is not invariant under conformal mappings (in fact, the only C^2 diffeomorphisms $g : \mathbf{R}^2 \to \mathbf{R}^2$ which preserve it are the affine mappings). Hence (6.4) cannot be the same as removability. But this did not tell us which of the implications is false. Maybe they are both. Jones and Murai showed in [JM] that there exist non-removable sets for which (6.4) holds, but the other direction is unknown. Another example of Jones–Murai type using the Menger curvature was given by Joyce and Mörters in [JoM].

7 Sets with infinite $\mathcal{H}^1$ measure

Let us now see what we know and what we don't know about removable sets after David's theorem. We have the full geometric characterization for compact sets E with $\mathcal{H}^1(E) < \infty$. We also know that sets with Hausdorff dimension bigger that one are not removable. Thus the only problematic sets E left are those with Hausdorff dimension one and $\mathcal{H}^1(E) = \infty$. It is easy to give examples of them by modifying the construction of the Cantor set C_1. Instead of taking the ratio of the side-lengths to be 1/4 at each step, let it be λ_k at the stage k. If $\lambda_k > 1/4$ and $\lambda_k \to 1/4$, then the Hausdorff dimension of the resulting Cantor set $C(\lambda)$ related to the sequence $\lambda = (\lambda_k)$ is one. If λ_k tends to 1/4 sufficiently quickly, then $\mathcal{H}^1\big(C(\lambda)\big) < \infty$, and $C(\lambda)$ is removable, by the same argument as for C_1. The exact condition for this is

$$\sup_n 4^n \sigma_n < \infty$$

where

$$\sigma_n = \lambda_1 \dots \lambda_n = \text{the side-length of } Q_{n,i}.$$

Let μ be the natural uniformly distributed measure on $C(\lambda)$. This means that $\mu(Q_{k,i}) = 4^{-k}$ for every square $Q_{k,i}$ of the generation k. If $\lambda_k \to 1/4$ sufficiently slowly,

$$\int \frac{1}{|\zeta - z|}\, d\mu\zeta \tag{7.1}$$

is bounded, and consequently also the Cauchy transform C_μ of μ. Thus E is not removable. The exact condition for the boundedness of (7.1) is

$$\sum_n \frac{1}{4^n \sigma_n} < \infty.$$

The sequences which are between these two cases are rather problematic and it has been possible to deal with them only with the Menger curvature methods. It is still rather easy to show that for μ as above (see [M3]) that

$$\iiint c(x,y,z)^2\, d\mu x\, d\mu y\, d\mu z < \infty \tag{7.2}$$

if and only if

$$\sum_n \frac{1}{(4^n \sigma_n)^2} < \infty. \tag{7.3}$$

Somewhat more generally, see [E], one can show that if (7.3) fails, there is no Borel measure μ on $C(\lambda)$ with linear growth; $\mu\big(B(x,r)\big) \leq cr$, which would satisfy (7.2).

We can now conclude from the equivalence of (7.2) and (7.3) and the following general result of Melnikov in [Me1] that $C(\lambda)$ is not removable if (7.3) holds.

Theorem 7.1 *Let $E \subset \mathbf{C}$ be compact. If there exists a non-negative Borel measure μ such that $\mu(E) > 0$, $\mu(\mathbf{C} \setminus E) = 0$,*

$$\mu\big(B(z,r)\big) \leq r \qquad \text{for } z \in \mathbf{C},\ r > 0, \qquad \text{and} \tag{1}$$

$$\iiint c(x,y,z)^2\, d\mu x\, d\mu y\, d\mu z < \infty, \tag{2}$$

then E is not removable.

The proof of Theorem 7.1 is a little similar in spirit to the proof of Theorem 3.1 as discussed in Section 4. By the computations of Section 5, the condition (7.2) is related to L^2-boundedness which by duality arguments leads to the existence of non-constant bounded analytic functions in $\mathbf{C} \setminus E$.

Melnikov conjectured that also the converse holds: the non-removability of E implies the existence of a measure μ with above properties. For the Cantor sets $C(\lambda)$ such a measure exists if and only if (7.3) holds. So to verify Melnikov's conjecture for these sets we have left to show that if $C(\lambda)$ is not removable then (7.3) holds. Mateu, Tolsa and Verdera proved this in [MTV]. Let us say a few words about the proof. We start again with a non-constant bounded analytic function $f : \mathbf{C} \setminus C(\lambda) \to \mathbf{C}$ and show that its existence implies that (7.3) holds. In the case of finite $\mathcal{H}^1$ measure we quickly got from this much more information, in particular, the representation formula (2.3) and even (2.4). Now the situation is much worse. There are bounded analytic functions which cannot be represented as a Cauchy transform of any complex measure. But one can work with approximations of $C(\lambda)$ which have finite length, for example, with the unions of the boundaries of the 4^n squares of side-length σ_n which appear in the construction of $C(\lambda)$. One does not prove directly (7.3) but as in the case of finite $\mathcal{H}^1$ measure, one proves L^2-boundedness using $T(b)$-theorems, in particular the form proved by Nazarov, Treil and Volberg in [NTV]. Then one uses the relations between Cauchy kernel and Menger curvature as before to get (7.3) from (7.2). The big problem is now to control the constants involved. When $\mathcal{H}^1\big(C(\lambda)\big) = \infty$ one needs good new ideas to prevent them from blowing up when closer and closer approximations of $C(\lambda)$ are used.

The natural question now is whether Melnikov's conjecture holds generally. It does. This has been proven by Tolsa in [T1], and it ends the long search for a geometric characterization of removable sets of bounded analytic functions. It can be argued that the existence of such a measure is not really a geometric condition, but at least it is not complex analytic, and any such characterization has been missing. It is not always easy to verify this condition, that is, to construct the measure. A nice test case is that of compact connected sets with more than one point. They are not removable by Riemann's mapping theorem, so such a measure exists and Jones has shown how to construct it. But it is not very easy, see [P1] for this.

Let us still formulate Tolsa's theorem:

Theorem 7.2 *Let $E \subset \mathbf{C}$ be compact. Then E is not removable if and only if there exists a non-negative Borel measure μ such that $\mu(E) > 0$, $\mu(\mathbf{C} \setminus E) = 0$,*

$$\begin{array}{ll} \mu\big(B(z,r)\big) \leq r \qquad \text{for } z \in C,\ r > 0, & \text{and} \\ \iiint c(x,y,z)^2 \, d\mu x \, d\mu y \, d\mu z < \infty. & \end{array}$$

The proof follows similar general lines as explained before for the Cantor sets $C(\lambda)$. That is it involves approximations of E with sets with finite $\mathcal{H}^1$ measure and application of the $T(b)$-theorem of [NTV].

To show how hugely this result adds to our understanding of the removable sets, let us look at their invariance under mappings. Before this it was not known if they are preserved under affine bijections of the plane. For example, if E is removable, is

$\{(x, 2y) : (x, y) \in E\}$ also removable? It is clear that Theorem 7.2 answers this in the positive. But it does much more. It is not very difficult to show from this that removability is preserved under $C^{1+\varepsilon}$ diffeomorphisms. It is much more difficult, but Tolsa has done it recently in [T2] to show that it is also preserved under bilipschitz mappings.

Another consequence of Theorem 7.2, or rather a quantitative form of it, is the semiadditivity of analytic capacity: there exists an absolute constant C such that for all compact sets $E_1, E_2, \dots \subset \mathbf{C}$,

$$\gamma\left(\bigcup_{i=1}^{\infty} E_i\right) \leq C \sum_{i=1}^{\infty} \gamma(E_i).$$

It is not known if this holds with $C = 1$.

Received: March 2003. Revised: June 2003.

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