Exceptional Points and Dynamical Phase Transitions

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Abstract

In the framework of non-Hermitian quantum physics, the relation between exceptional points, dynamical phase transitions and the counterintuitive behavior of quantum systems at high level density is considered. The theoretical results obtained for open quantum systems and proven experimentally some years ago on a microwave cavity, may explain environmentally induced effects (including dynamical phase transitions), which have been observed in various experimental studies. They also agree (qualitatively) with the experimental results reported recently in $\mathcal{PT}$ symmetric optical lattices.

Keywords: non-Hermitian quantum physics, dynamical phase transitions, exceptional points, open quantum systems, PT-symmetric optical lattices.

Many years ago, Kato [1] introduced the notation *exceptional points* for singularities appearing in the perturbation theory for linear operators. Consider a family of operators of the form

$$T(\varsigma) = T(0) + \varsigma T' \tag{1}$$

where ς is a scalar parameter, $T(0)$ is the unperturbed operator and $\varsigma T'$ is the perturbation. Then the number of eigenvalues of $T(\varsigma)$ is independent of ς with the exception of some special values of ς where (at least) two eigenvalues coalesce. These special values of ς are called exceptional points. An example is the operator

$$T(\varsigma) = \begin{pmatrix} 1 & \varsigma \\ \varsigma & -1 \end{pmatrix} . \tag{2}$$

In this case, the two values $\varsigma = \pm$ i give the same eigenvalue 0. According to Kato, not only the number of eigenvalues but also the number of eigenfunctions is reduced at the exceptional point.

Operators of the type (2) appear in the description of physical systems, for example in the theory of open quantum systems [2]. Here, the function space of the system consisting of well localized states is embedded in an extended environment of scattering wavefunctions. Due to this embedding, the Hamiltonian of the system is non-Hermitian. The interaction between two neighboring levels is given by a 2×2 symmetric Hamiltonian that describes the two-level system with the unperturbed energies ϵ_1 and ϵ_2 and the interaction ω between the two levels,

$$H(\omega) = \begin{pmatrix} \epsilon_1 & \omega \\ \omega & \epsilon_2 \end{pmatrix} . \tag{3}$$

The operators (2) and (3) are, indeed, of the same type.

In the following, we will discuss the role played by exceptional points in physical systems. It will be shown that they influence not only resonance states but also discrete states lying beyond the energy window coupled directly to the environment. Furthermore (and most important), they are responsible for the appearance of dynamical phase transitions occurring in the regime of overlapping resonances.

In an open quantum system, two states can interact directly (corresponding to a first-order term) as well as via an environment (described by a second-order term) [2]. Here, we consider the case that the direct interaction is contained in the energies ϵ_k ($k = 1, 2$). This means that the ϵ_k are considered to be eigenvalues of a non-Hermitian Hamilton operator H_0 which contains both the direct interaction V between the two states and also the coupling of each of the two individual states to the environment of scattering wavefunctions. Then ω contains exclusively the coupling of the two states via the environment. This allows one to study environmentally induced effects in open quantum systems in a very clear manner.

The eigenvalues of the operator $H(\omega)$ are

$$\varepsilon_{1,2} = \frac{\epsilon_1 + \epsilon_2}{2} \pm Z \; ; \; Z = \frac{1}{2}\sqrt{(\epsilon_1 - \epsilon_2)^2 + 4\omega^2} \, . \tag{4}$$

The physical meaning of $\mathrm{Re}(Z)$ is the well-known *level repulsion* occurring at small (mostly real) ω while $\mathrm{Im}(Z)$ is related to *width bifurcation.* The two eigenvalue trajectories cross when $Z = 0$, i.e. when

$$\frac{\epsilon_1 - \epsilon_2}{2\omega} = \pm i \, . \tag{5}$$

At these *crossing points*, the two eigenvalues coalesce,

$$\varepsilon_1 = \varepsilon_2 \equiv \varepsilon_0 \, . \tag{6}$$

The crossing points may therefore be called exceptional points. They have a nontrivial topological structure. For details and for a reference to the experimental proof, see the review [2].

The eigenfunctions of the non-Hermitian operator (3) are biorthogonal,

$$\langle \phi_k^* | \phi_l \rangle = \delta_{k,l} \; . \tag{7}$$

When the distance between the two individual states is large and they do not overlap, they are almost orthogonal in the standard manner, $\langle \phi_k^* | \phi_l \rangle \approx \langle \phi_k | \phi_l \rangle$. In approaching the exceptional point, however, they become linearly dependent,

$$\phi_1^{\rm cr} \to \; \pm \, i \; \phi_2^{\rm cr} \qquad\qquad \phi_2^{\rm cr} \to \mp i \, \phi_1^{\rm cr} \; . \tag{8}$$

Hence, the phases of the eigenfunctions ϕ_k of the non-Hermitian Hamilton operator (3) are not rigid. A quantitative measure for *phase rigidity* is the value

$$r_k \equiv \frac{\langle \phi_k^* | \phi_k \rangle}{\langle \phi_k | \phi_k \rangle} \tag{9}$$

which varies between 1 at large distance of the states and 0 at the exceptional point. Further details, including the experimental proof of relations (8) and (9), are discussed in the review [2].

One of the most interesting differences between Hermitian and non-Hermitian quantum physics is surely the fact that the phases of the eigenfunctions of the Hamiltonian are rigid ($r_k = 1$) in the first case while they may vary according to (9) in the second case [2]. It is possible therefore that the wavefunction of one of the two states aligns with the scattering wavefunction of the environment while the other state decouples (more or less) from the environment. This phenomenon, called *resonance trapping*, is caused by $\mathrm{Im}(Z)$ in (4), i.e. by width bifurcation. It starts near (or at) the crossing (exceptional) point under the influence of the continuum of scattering wavefunctions. This means that the non-Hermitian quantum physics is able to describe environmentally induced effects, for example spectroscopic redistribution processes induced by the mixing of the states via the continuum of scattering wavefunctions, which is described by ω in (3).

Another feature involved in non-Hermitian quantum physics is the appearance of nonlinearities in the neighborhood of exceptional points [2]. For example, the S matrix at a double pole (corresponding to an exceptional point) in the two-level one-continuum case reads

$$S = 1 - 2i \frac{\Gamma_0}{E - E_0 + \frac{i}{2}\Gamma_0} - \frac{\Gamma_0^2}{(E - E_0 + \frac{i}{2}\Gamma_0)^2} \tag{10}$$

where the notation (6) is used and $\varepsilon_0 \equiv E_0 - \dfrac{i}{2}\Gamma_0$. At the exceptional point, the cross section vanishes due to interferences. The minimum is washed out in the neighborhood of the double pole, however, the resonance is broader than a Breit-Wigner resonance according to (10).

Further studies have shown that the effects discussed above by means of the toy model (3) survive when the full problem in the whole function space with many levels is considered. This means that, when the level density is high and the individual resonances overlap, Hamilton operators of type (3) and the exceptional points related to them play an important role for the dynamics of the system. Mainly two types of phenomena are caused by the exceptional points in physical systems. The two phenomena condition each other (for details see [2]).

First, the *spectroscopy of discrete and resonance states* is strongly influenced by exceptional points. Both types of states are eigenstates of a non-Hermitian many-level Hamilton operator being analogous to (3). They differ by the boundary conditions. The states are discrete (corresponding to an infinitely long lifetime) when their energy is beyond the window coupled to the continuum of scattering wavefunctions. The states are resonant (corresponding, in general, to a finite lifetime) when their energy is inside the window coupled to the continuum of scattering wavefunctions. Accordingly, the exceptional points influence the behavior not only of the resonance states but also of the discrete states. For example, the avoided crossing of discrete states can be traced back to an exceptional point and, furthermore, the mixing of discrete states around an avoided crossing of levels is shown to arise from the existence of an exceptional point in the continuum of scattering wavefunctions.

Discrete states have been well described in the framework of conventional quantum mechanics for very many years. The Hamiltonian is Hermitian with effective forces that are not calculated in the standard theory. The effective forces simulate (at least partly) the principal value integral which arises from the coupling to other states via the continuum (denoted by ω in (3)). The phases of the eigenfunctions are rigid ($r_k = 1$), the discrete states avoid crossing and the topological phase of the diabolic point is the Berry phase. Due to $r_k = 1$, the Schrödinger equation is linear, and the levels are mixed (entangled) in the whole parameter range of avoided level crossings. At the critical point, the mixing is maximal $(1 : 1)$.

Resonance states are well described when quantum theory is extended by including the environment of scattering wavefunctions into the formalism. The Hamiltonian is, in general, non-Hermitian and ω in (3) is complex since it contains both the principal value integral and the residuum arising from the coupling to other states via the continuum. The phases of the eigenfunctions are, in general, not rigid corresponding to (9) with $0 \le r_k \le 1$. This can be seen in the *skewness* of the basis. The resonance states can cross in the continuum (at the exceptional point) and the topological phase of the crossing point is twice the Berry phase. When $r_k < 1$ (regime of resonance overlapping with avoided level crossings), the Schrödinger equation is nonlinear and the levels are mixed (entangled) in the parameter range in which the resonances overlap. The parameter range shrinks to one point when the levels cross, i.e. when $r_k \to 0$ and (8) is approached.

Secondly, a *dynamical phase transition* is induced by exceptional points in the regime of overlapping resonances. Such a phase transition is environmentally induced and occurs due to width bifurcation. The number of localized states is reduced since a few resonance states align to the scattering states of the environment and cease to be localized. By this, the dynamical phase transition destroys the relation between localized states below and above the critical regime in which the resonances overlap.

The two phases are characterized by the following properties. In one of the phases, the discrete and narrow resonance states have individual spectroscopic features. Here, the real parts (energies) of the eigenvalue trajectories avoid crossing while the imaginary parts (widths) can cross. As a function of increasing (but small) coupling strength between system and environment, the number of localized states does not change and the widths of the resonance states increase, as expected. Here, the exceptional points are of minor importance.

In the other phase, the narrow resonance states are superimposed with a smooth background and the individual spectroscopic features are lost. The narrow resonance states appear due to resonance trapping, i.e. as a consequence of the alignment of a small number of resonance states to the environment (for details see [2]). Here, the real parts (energies) of the eigenvalue trajectories of narrow (trapped) resonance states can cross with those of the broad (aligned) states since they exist at different time scales. The narrow resonance states show a counterintuitive behavior: with increasing (strong) coupling strength between system and environment, the widths of the narrow (trapped) states decrease. Furthermore, the number of trapped resonance states is smaller than the number of individual (basic) states. This means, that the number of localized states is reduced when the (complex) interaction ω in (3) is sufficiently large. This phase results from the spectroscopic redistribution processes caused by exceptional points.

The transition region between the two phases is the regime of overlapping resonances. Here, short-lived and long-lived resonance states coexist, i.e. they are not clearly separated from one another in the time scale. In nuclear physics, this regime is described well by the *doorway picture*. According to this picture, the long-lived states are decoupled from the continuum while the doorway states are coupled to both the continuum *and* the long-lived states. In the transition region, the cross section is enhanced due to the (partial) alignment of some states (of the doorway states) with the scattering states of the environment.

It is interesting to see that the system behaves according to expectations only at low level density. Here, the resonance states are characterized by their individual spectroscopic properties and their number does not change by varying a parameter. After passing the transition regime with overlapping resonances by further variation of the parameter, the behavior of the system becomes counterintuitive: the narrow resonance states decouple more or less from the continuum of scattering wavefunctions and the number of localized states decreases. The decoupling increases with increasing coupling strength between system and environment. This counterintuitive behavior was proven experimentally, some years ago, in a study on a microwave cavity [3].

Recently, a dynamical phase transition and the counterintuitive behavior at strong coupling between system and environment has been observed experimentally also in $\mathcal{PT}$ symmetric optical lattices. At small loss, the transmission through the system decreases with increasing loss according to expectations. With further increasing loss, however, the $\mathcal{PT}$ symmetry breaks and the transmission is enhanced [4, 5, 6]. An interpretation of these results from the point of view of a dynamical phase transition can be found in [7]. Here, also the difference between the discrete states in $\mathcal{PT}$ symmetric systems and the *bound states in the continuum* (resonance states with vanishing decay width) in open symmetric quantum systems with overlapping resonances is discussed.

Dynamical phase transitions in other systems are observed experimentally. They are discussed in [2, 7, 8]. Common to all of them is that the dynamical phase transition takes place in the regime of overlapping resonances. As a result, a few states are aligned to the scattering states of the environment while the remaining ones (long-lived trapped resonance states) are (almost) decoupled from the continuum of scattering wavefunctions.

The results discussed in the present paper can be summarized as follows. Exceptional points play an important role in the dynamics of quantum systems. They are responsible, e.g., for the appearance of dynamical phase transitions in the regime of overlapping resonances. In approaching exceptional points by varying a parameter, some states align with the states of the environment by trapping almost all the other resonance states. Such a process can be described in non-Hermitian quantum physics since the phases of the eigenfunctions of the Hamiltonian are not rigid, $1 \geq r_k \geq 0$. However, it cannot be described in conventional Hermitian quantum theory with fixed phases of the eigenfunctions, $r_k = 1$. Due to the alignment of some states to the states of the environment, physical processes such as transmission may be enhanced in a comparably large parameter range. The alignment increases with increasing coupling strength between system and environment and causes a behavior of the system at high level density which is counterintuitive at first glance. Further theoretical and experimental studies in this field will broaden our understanding of quantum mechanics. Moreover, the results are expected to be of great value for applications.

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