

Geometrical properties of the space of idempotent probability measures

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ABSTRACT

Although traditional and idempotent mathematics are “parallel”, by an application of the category theory we show that objects obtained the similar rules over traditional and idempotent mathematics must not be “parallel”. At first we establish for a compact metric space X the spaces $P(X)$ of probability measures and $I(X)$ idempotent probability measures are homeomorphic (“parallelism”). Then we construct an example which shows that the constructions P and I form distinguished functors from each other (“parallelism” negation). Further for a compact Hausdorff space X we establish that the hereditary normality of $I_3(X) \setminus X$ implies the metrizable of X .

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1. INTRODUCTION

Idempotent mathematics is a new branch of mathematical sciences, rapidly developing and gaining popularity over the last two decades. It is closely related to mathematical physics. The literature on the subject is vast and includes numerous books and an all but innumerable body of journal papers. An important stage of development of the subject was presented in the book

“Idempotency” edited by J. Gunawardena [6]. This book arose out of the well-known international workshop that was held in Bristol, England, in October 1994.

The next stage of development of idempotent and tropical mathematics was presented in the book *Idempotent Mathematics and Mathematical Physics* edited by G. L. Litvinov and V. P. Maslov [13]. The book arose out of the international workshop that was held in Vienna, Austria, in February 2003. In [14] it was delivered the proceedings of the International Workshop on Idempotent and Tropical Mathematics and Problems of Mathematical Physics, held at the Independent University of Moscow, Russia, on August 25-30, 2007.

Idempotent mathematics is based on replacing the usual arithmetic operations with a new set of basic operations, i. e., on replacing numerical fields by idempotent semirings and semifields. Typical example is the so-called max-plus algebra \mathbb{R}_{\max} [8], [19].

The modern idempotent analysis (or idempotent calculus, or idempotent mathematics) was founded by V. P. Maslov and his collaborators [11], [12], [10], [14]. Some preliminary results are due to E. Hopf and G. Choquet, see [1], [7].

Idempotent mathematics can be treated as the result of a dequantization of the traditional mathematics over numerical fields as the Planck constant h tends to zero taking imaginary values. This point of view was presented in [13]. In other words, idempotent mathematics is an asymptotic version of the traditional mathematics over the fields of real and complex numbers.

The basic paradigm is expressed in terms of an idempotent correspondence principle. This principle is closely related to the well-known correspondence principle of N. Bohr in quantum theory. Actually, there exists a heuristic correspondence between important, interesting, and useful constructions and results of the traditional mathematics over fields and analogous constructions and results over idempotent semirings and semifields (i. e., semirings and semifields with idempotent addition).

A systematic and consistent application of the idempotent correspondence principle leads to a variety of results, often quite unexpected. As a result, in parallel with the traditional mathematics over fields, its “shadow,” idempotent mathematics, appears. This “shadow” stands approximately in the same relation to traditional mathematics as classical physics does to quantum theory.

The notion of idempotent (Maslov) measure finds important applications in different parts of mathematics, mathematical physics and economics (see the survey article [14] and the bibliography therein). Topological and categorical properties of the functor of idempotent measures were studied in [20], [21]. Although idempotent measures are not additive and the corresponding functionals are not linear, there are some parallels between topological properties of the functor of probability measures and the functor of idempotent measures (see, for example [16], [17], [20]) which are based on existence of natural equiconnectedness structure on both functors.

A notion of central importance in categorical topology is that of topological functor. Various applications of topological functors described in [5].

In the present paper we show that for a compact metric space X the spaces $P(X)$ of probability measures and $I(X)$ idempotent probability measures are homeomorphic. Further we construct an example which shows that the constructions P and I form distinguished functors from each other. This phenomenon shows that the category theory finds out such subtle moments of relations between topological spaces which against common sense. In other words, we get such a conclusion:

Although traditional and idempotent mathematics are parallel, an application of the category theory shows, objects obtained the similar rules over traditional and idempotent mathematics must not be “parallel”.

Further, for a compact Hausdorff space X we establish that the hereditary normality of $I_3(X) \setminus X$ implies the metrizability of X .

2. PRELIMINARIES

Recall [14] that a set S equipped with two algebraic operations: addition \oplus and multiplication \odot , is said to be a *semiring* if the following conditions are satisfied:

- the addition \oplus and the multiplication \odot are associative;
- the addition \oplus is commutative;
- the multiplication \odot is distributive with respect to the addition \oplus :

$$\begin{aligned} x \odot (y \oplus z) &= x \odot y \oplus x \odot z \quad \text{and} \\ (x \oplus y) \odot z &= x \odot z \oplus y \odot z \end{aligned}$$

for all $x, y, z \in S$.

A *unit* of a semiring S is an element $\mathbf{1} \in S$ such that $\mathbf{1} \odot x = x \odot \mathbf{1} = x$ for all $x \in S$. A *zero* of the semiring S is an element $\mathbf{0} \in S$ such that $\mathbf{0} \neq \mathbf{1}$ and $\mathbf{0} \oplus x = x \oplus \mathbf{0} = x$ for all $x \in S$. A semiring S with neutral elements $\mathbf{0}$ and $\mathbf{1}$ is called a *semifield* if every nonzero element of S is invertible.

A semiring S is called an *idempotent semiring* if $x \oplus x = x$ for all $x \in S$. An idempotent semiring S is called an *idempotent semifield* if it is a semifield. Note that dioïds, quantales and inclines are examples of idempotent semirings [14]. Many authors (S. C. Kleene, S. N. N. Pandit, N. N. Vorobjev, B. A. Carré, R. A. Cuninghame-Green, K. Zimmermann, U. Zimmermann, M. Gondran, F. L. Baccelli, G. Cohen, S. Gaubert, G. J. Olsder, J.-P. Quadrat, V. N. Kolokoltsov and others) used idempotent semirings and matrices over these semirings for solving some applied problems in computer science and discrete mathematics.

Let $\mathbb{R} = (-\infty, +\infty)$ be the field of real numbers and $\mathbb{R}_+ = [0, +\infty)$ be the semiring of all nonnegative real numbers (with respect to the usual addition “+” and multiplication “.”). Consider a map $\Phi_h: \mathbb{R}_+ \rightarrow \mathbb{R}^{(h)} = \mathbb{R} \cup \{-\infty\}$

defined by the equality

$$\Phi_h(x) = h \ln x, \quad h > 0.$$

Let $x, y \in X$ and $u = \Phi_h(x), v = \Phi_h(y)$. Put $u \oplus_h v = \Phi_h(x + y)$ and $u \odot v = \Phi_h(xy)$. The image $\Phi_h(0) = -\infty$ of the usual zero 0 is a zero $\mathbf{0}$ and the image $\Phi_h(1) = 0$ of the usual unit 1 is a unit $\mathbf{1}$ in S with respect to these new operations. The convention $-\infty \odot x = -\infty$ allows us to extend \oplus_h and \odot over $\mathbb{R}^{(h)}$. Thus we obtained the structure of a semiring $(\mathbb{R}^{(h)}, \oplus_h, \odot)$ which is isomorphic to $(\mathbb{R}_+, +, \cdot)$.

A direct check shows that $u \oplus_h v \rightarrow \max\{u, v\}$ as $h \rightarrow 0$. It can easily be checked that $\mathbb{R} \cup \{-\infty\}$ forms a semiring with respect to the addition $u \oplus v = \max\{u, v\}$ and the multiplication $u \odot v = u + v$ with zero $\mathbf{0} = -\infty$ and unit $\mathbf{1} = 0$. Denote this semiring by \mathbb{R}_{\max} ; it is idempotent, i. e., $u \oplus u = u$ for all its elements u . The semiring $(\mathbb{R}_{\max}, \oplus, \odot)$ generates the semifield $(\mathbb{R}_{\max}, \oplus, \odot, \mathbf{0}, \mathbf{1})$. The analogy with quantization is obvious; the parameter h plays the role of the Planck constant, so \mathbb{R}_+ can be viewed as a “quantum object” and \mathbb{R}_{\max} as the result of its “dequantization”. The described passage $(\mathbb{R}_+, +, \cdot, 0, 1) \xrightarrow{\Phi_h} (\mathbb{R}^{(h)}, \oplus_h, \odot, -\infty, 0) \xrightarrow{h \rightarrow 0} (\mathbb{R}_{\max}, \oplus, \odot, \mathbf{0}, \mathbf{1})$ is called the *Maslov dequantization*.

Let X be a compact Hausdorff space, $C(X)$ be the algebra of continuous functions on X with the usual algebraic operations (i. e. with the addition “+” and the multiplication “·”). On $C(X)$ the operations \oplus and \odot are determined by $\varphi \oplus \psi = \max\{\varphi, \psi\}$ and $\varphi \odot \psi = \varphi + \psi$ where $\varphi, \psi \in C(X)$.

Recall [21] that a functional $\mu: C(X) \rightarrow \mathbb{R}$ is said to be an *idempotent probability measure* on X if it has the following properties:

- (I1) $\mu(\lambda_X) = \lambda$ for all $\lambda \in \mathbb{R}$, where λ_X is a constant function;
- (I2) $\mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi)$ for all $\lambda \in \mathbb{R}$ and $\varphi \in C(X)$;
- (I3) $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$ for all $\varphi, \psi \in C(X)$.

Let $I(X)$ denote the set of all idempotent probability measures on a compact Hausdorff space X , and $\mathbb{R}^{C(X)}$ be a set of all maps $C(X) \rightarrow \mathbb{R}$. Obviously $I(X) \subset \mathbb{R}^{C(X)}$. One can treat $\mathbb{R}^{C(X)} = \prod_{\varphi \in C(X)} \mathbb{R}_\varphi$ where $\mathbb{R}_\varphi = \mathbb{R}, \varphi \in C(X)$.

We consider $\mathbb{R}^{C(X)}$ with the product topology and consider $I(X)$ as its subspace. A family of sets of the form

$$\langle \mu; \varphi_1, \dots, \varphi_n; \varepsilon \rangle = \{ \nu \in I(X) : |\nu(\varphi_i) - \mu(\varphi_i)| < \varepsilon, i = 1, \dots, n \}$$

is a base of open neighbourhoods of a given idempotent probability measure $\mu \in I(X)$ according to the induced topology, where $\varphi_i \in C(X), i = 1, \dots, n$, and $\varepsilon > 0$. It is obvious that the induced topology and the pointwise convergence topology on $I(X)$ coincide. So, we get a topological space $I(X)$, equipped with the pointwise convergence topology. In [21] it was shown that for each compact Hausdorff space X the space $I(X)$ is also a compact Hausdorff space.

Let X, Y be compact Hausdorff spaces and $f: X \rightarrow Y$ be a continuous map. It is easy to check that the map $I(f): I(X) \rightarrow I(Y)$ determined by the formula $I(f)(\mu)(\psi) = \mu(\psi \circ f)$ is continuous. The construction I is a normal functor

acting in the category \mathbf{Comp} of compact Hausdorff spaces and their continuous maps.

Remind that a functor $F: \mathbf{Comp} \rightarrow \mathbf{Comp}$ on the category of compact Hausdorff spaces and continuous maps is said to be normal (see [15], Definition 14) if it satisfies the following conditions:

- (F1) F is continuous (i. e., $F(\lim S) = \lim F(S)$);
- (F2) F preserves weight (i. e., $wX = wF(X)$);
- (F3) F is monomorphic (i. e., preserves the injectivity of maps);
- (F4) F is epimorphic (i. e., preserves the surjectivity of maps);
- (F5) F preserves intersections (i. e., $F(\bigcap_{\alpha} X_{\alpha}) = \bigcap_{\alpha} F(X_{\alpha})$);
- (F6) F preserves preimages (i. e., $F(f^{-1}) = F(f)^{-1}$);
- (F7) F preserves singletons and the empty space (i. e., $F(1) = 1$ and $F(\emptyset) = \emptyset$).

Let us decipher this definition. Let $S = \{X_{\alpha}, p_{\alpha}^{\beta}; \mathfrak{A}\}$ be an inverse system of compact Hausdorff spaces, and let $\lim S = \lim_{\leftarrow} S$ be its limit. According to the Kurosh theorem, the limit of any inverse system of nonempty compact Hausdorff spaces is nonempty (see [2], Theorem 3.13) and compact Hausdorff space (see [2], Proposition 3.12). The action of the functor F on the compact Hausdorff spaces X_{α} and the maps p_{α}^{β} , where $\alpha, \beta \in \mathfrak{A}$ and $\alpha \prec \beta$, produces the inverse system

$$F(S) = \{F(X_{\alpha}), F(p_{\alpha}^{\beta}); \mathfrak{A}\}.$$

Let $\lim F(S)$ be the limit of this system. By virtue of condition (F1), we have $F(\lim S) = \lim F(S)$. Given a topological space X , let wX denote its weight, i. e., the minimum cardinality of a base of X . By condition (F2), the weights of the compact spaces X and $F(X)$ are equal. Since the functor F is monomorphic (by condition (F3)), we can assume $F(A)$ to be a subspace of $F(X)$ for a closed $A \subset X$. The space $F(A)$ is identified with a subspace of $F(X)$ by means of the embedding $F(i_A)$, where $i_A: A \rightarrow X$ is the identity embedding. According to condition (F4), if $f: X \rightarrow Y$ is a continuous map “onto” then so is $F(f): F(X) \rightarrow F(Y)$. For a monomorphic functor F , conditions (F5) and (F6) mean that, for any family $\{A_{\alpha}\}$ of closed subsets of a compact Hausdorff space X , we have

$$F(\bigcap_{\alpha} A_{\alpha}) = \bigcap_{\alpha} F(A_{\alpha})$$

(this is condition (F5)), and for any continuous map $f: X \rightarrow Y$ and any closed B in Y , we have

$$F(f^{-1}(B)) = F(f)^{-1}(F(B))$$

(this is condition (F6)). The singleton preservation condition means that F takes any one-point space to a one-point space.

The intersection preservation condition makes it possible to define an important notion of the support of a monomorphic functor F . The support of a point $x \in F(X)$ is a closed set $\text{supp } x \subset X$ such that, for any closed $A \subset X$,

we have $A \supset \text{supp } x$ if and only if $x \in F(A)$ ([15], Definition 18). Given an intersection-preserving functor F , each point $x \in F(X)$ has support, which is defined by

$$\text{supp } x = \cap \{A \subset X : \overline{A} = A, x \in F(A)\},$$

where \overline{A} denotes the closure of A .

As it was mentioned above, the functor I is normal; therefore, for each compact Hausdorff space X and any idempotent probability measure $\mu \in I(X)$, the support of μ is defined as:

$$\text{supp } \mu = \cap \{A \subset X : \overline{A} = A, \mu \in I(A)\}.$$

For a positive integer n we define the following set

$$I_n(X) = \{\mu \in I(X) : |\text{supp } \mu| \leq n\}.$$

Put

$$I_\omega(X) = \bigcup_{n=1}^{\infty} I_n(X).$$

The set $I_\omega(X)$ is everywhere dense in $I(X)$ [18], [21]. An idempotent probability measure $\mu \in I_\omega(X)$ is called an idempotent probability measure with finite support. Note that if μ is an idempotent probability measure with the finite support $\text{supp } \mu = \{x_1, x_2, \dots, x_k\}$ then μ can be represented as $\mu = \lambda_1 \odot \delta_{x_1} \oplus \lambda_2 \odot \delta_{x_2} \oplus \dots \oplus \lambda_k \odot \delta_{x_k}$ uniquely, where $-\infty < \lambda_i \leq 0$, $i = 1, \dots, k$, $\lambda_1 \oplus \lambda_2 \oplus \dots \oplus \lambda_k = 0$. Here, as usual, for $x \in X$ by δ_x we denote a functional on $C(X)$ defined by the formula $\delta_x(\varphi) = \varphi(x)$, $\varphi \in C(X)$, and called the Dirac measure. It is supported at the point x .

Let X be a compact Hausdorff space. A continuous linear functional $\mu: C(X) \rightarrow \mathbb{R}$ is said to be a *measure* on X . The Riesz theorem about isomorphism between the normalized space $(C(X))^*$ dual to $C(X)$ (i. e. the space of all continuous functional on $C(X)$) and the space $M(X)$ of all finite regular measures on X is substantiation of the above definition (see [4], page 192, paragraph 3.1).

A measure $\mu \in M(X)$ is *positive* ($\mu \geq 0$) if $\mu(\varphi) \geq 0$ for each $\varphi \in C(X)$, $\varphi \geq 0$.

A measure μ is positive if and only if $\|\mu\| = \mu(1_X)$.

Really, let $\mu \geq 0$ and $\|\varphi\| \leq 1$. Then $\mu(1_X - \varphi) \geq 0$. Consequently, $\mu(1_X) \geq \mu(\varphi)$. From here $\|\mu\| = \sup\{\mu(\varphi) : \varphi \in C(X), \|\varphi\| \leq 1\} = \mu(1_X)$.

Contrary, let now $\mu(1_X) = \|\mu\|$ and $\varphi \geq 0$. Put $\psi = 1_X - \frac{\varphi}{\|\varphi\|}$. Since $\|\psi\| \leq 1$ we have $\mu(\psi) \leq \|\mu\| = \mu(1_X)$, i. e. $\mu(1_X) - \frac{1}{\|\varphi\|} \cdot \mu(\varphi) \leq \mu(1_X)$. Hence $\mu(\varphi) \geq 0$.

A measure μ is normed, if $\|\mu\| = 1$. A positive, normed measure is said to be a *probability measure*.

Thus, we can define the notion of probability measure as the following.

A *probability measure* on a given compact Hausdorff space X is a functional $\mu: C(X) \rightarrow \mathbb{R}$ satisfying the conditions:

(P1) $\mu(\lambda_X) = \lambda$ for all $\lambda \in \mathbb{R}$, where λ_X - constant function;

(P2) $\mu(\lambda\varphi) = \lambda\mu(\varphi)$ for all $\lambda \in \mathbb{R}$ and $\varphi \in C(X)$;

(P3) $\mu(\varphi + \psi) = \mu(\varphi) + \mu(\psi)$ for all $\varphi, \psi \in C(X)$.

The set of all probability measures on a compact Hausdorff space X is denoted by $P(X)$. The set $P(X)$ is endowed with the pointwise convergence topology, i. e. we consider $P(X)$ as a subspace of $\mathbb{R}^{C(X)}$.

It is well known the topological spaces $P(X)$ and $I(X)$ equipped with the pointwise convergence topology are compact Hausdorff spaces.

It is easy to see that, the conditions of normality (P1) and (I1) are the same, and the conditions of homogeneity (P2) and (I2), and the conditions of additivity (P3) and (I3) are mutually similar, just operations are different. In other words, the definition of the idempotent probability measure is “parallel” to the traditional one.

In section 3 we will show that the spaces $P(X)$ and $I(X)$ are homeomorphic, i. e. constructions P and I generate “parallel” objects. At the same time, in section 4 we will show that functors P and I are not isomorphic, i. e. the constructions P and I themselves are not “parallel”.

Note that idempotent probability measures were investigated in [21]. Unlike this work in the present paper we establish our results constructively, while in [21] the results were gotten descriptively.

3. SPACES $P(X)$ AND $I(X)$ ARE HOMEOMORPHIC

Theorem 3.1. *For an arbitrary Hausdorff finite space X the spaces $P(X)$ and $I(X)$ are homeomorphic.*

Proof. We determine the map

$$z_I^P : P(X) \longrightarrow I(X),$$

by the following equality

$$z_I^P \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) = \bigoplus_{i=1}^n \left(\left(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j \right) \odot \delta_{x_i} \right), \quad \sum_{i=1}^n \alpha_i \delta_{x_i} \in P(X),$$

here $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i > 0$ for all $i = 1, \dots, n$, and the map

$$z_P^I : I(X) \longrightarrow P(X),$$

by the rule

$$z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) = \sum_{i=1}^n \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} \cdot \delta_{x_i}, \quad \bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \in I(X),$$

here $\bigoplus_{i=1}^n \lambda_i = 0$, $\lambda_i > -\infty$, $i = 1, \dots, n$.

We will show that the maps z_I^P and z_P^I are continuous and mutually inverse.

1) For each probability measure $\sum_{i=1}^n \alpha_i \delta_{x_i} \in P(X)$ the following equalities hold

$$\begin{aligned} z_P^I \left(z_P^I \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) \right) &= z_P^I \left(\bigoplus_{i=1}^n \left(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j \right) \odot \delta_{x_i} \right) = \\ &= \sum_{i=1}^n \frac{e^{\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j}}{\sum_{l=1}^n e^{\ln \alpha_l - \bigoplus_{j=1}^n \ln \alpha_j}} \cdot \delta_{x_i} = \sum_{i=1}^n \frac{e^{\ln \alpha_i} : e^{\bigoplus_{j=1}^n \ln \alpha_j}}{\left(\sum_{l=1}^n e^{\ln \alpha_l} \right) : e^{\bigoplus_{j=1}^n \ln \alpha_j}} \cdot \delta_{x_i} = \\ &= \sum_{i=1}^n \frac{\alpha_i \delta_{x_i}}{\sum_{l=1}^n \alpha_l} = \sum_{i=1}^n \alpha_i \delta_{x_i}; \end{aligned}$$

2) For each idempotent probability measure $\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \in I(X)$ we have

$$\begin{aligned} z_P^I \left(z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) \right) &= z_P^I \left(\sum_{i=1}^n \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} \delta_{x_i} \right) = \\ &= \bigoplus_{i=1}^n \left(\ln \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} - \bigoplus_{l=1}^n \ln \frac{e^{\lambda_l}}{\sum_{j=1}^n e^{\lambda_j}} \right) \odot \delta_{x_i} = \\ &= \bigoplus_{i=1}^n \left(\ln e^{\lambda_i} - \ln \sum_{j=1}^n e^{\lambda_j} - \bigoplus_{l=1}^n \left(\ln e^{\lambda_l} - \ln \sum_{j=1}^n e^{\lambda_j} \right) \right) \odot \delta_{x_i} = \\ &= \bigoplus_{i=1}^n \left(\lambda_i - \ln \sum_{j=1}^n e^{\lambda_j} - \bigoplus_{l=1}^n \lambda_l + \ln \sum_{j=1}^n e^{\lambda_j} \right) \odot \delta_{x_i} = \bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i}. \end{aligned}$$

Consequently, the compositions $z_P^I z_P^I: I(X) \rightarrow I(X)$ and $z_P^I z_P^I: P(X) \rightarrow P(X)$ are the identical maps.

Now we will show that the maps z_P^I and z_P^I are continuous. Since they are mutually inverse maps between compact Hausdorff spaces, it suffices to show the continuity only of one of them.

We show that the map $z_P^I: P(X) \rightarrow I(X)$ is continuous. Let $\mu_0 = \sum_{i(0)=1}^{n_0} \alpha_0^{i(0)} \delta_{x_0^{i(0)}} \in P(X)$ be a probability measure, $\{\mu_t\}_{t=1}^\infty = \left\{ \sum_{i(t)=1}^{n_t} \alpha_t^{i(t)} \delta_{x_t^{i(t)}} \right\}_{t=1}^\infty \subset P(X)$ be a sequence converging to μ_0 in the pointwise convergence topology (symbolically $\lim_{t \rightarrow \infty} \mu_t = \mu_0$). It means that $\lim_{t \rightarrow \infty} \mu_t(\varphi) = \mu_0(\varphi)$ for all $\varphi \in C(X)$.

For each point $x_0^{i(0)} \in \text{supp } \mu_0 = \{x_0^1, x_0^2, \dots, x_0^{n_0}\}$ consider a characteristic function $\chi_{i(0)} = \chi_{\{x_0^{i(0)}\}} : X \rightarrow \mathbb{R}$, $i(0) = 1, \dots, n_0$. These functions are continuous, i. e. $\chi_{i(0)} \in C(X)$ since X is provided with the discrete topology. Evidently,

$$(\star) \quad \lim_{t \rightarrow \infty} \mu_t(\chi_{i(0)}) = \mu_0(\chi_{i(0)}) = \alpha_0^{i(0)}, \quad i(0) = 1, \dots, n_0.$$

(\star) implies the following two conclusions:

- (Output 1) Since each $\alpha_0^{i(0)} > 0$, we have $x_0^{i(0)} \in \text{supp } \mu_t$ for all t greater than or equals to some $t_{i(0)}$. Hence $\text{supp } \mu_0 \subset \text{supp } \mu_t$ for all $t \geq \max\{t_1, \dots, t_{n_0}\}$;
- (Output 2) Let $\alpha_t^{i(0)}$ be the barycentre mass of μ_t at $x_0^{i(0)} \in \text{supp } \mu_t$, $t = 1, 2, \dots$. Then $\lim_{t \rightarrow \infty} \alpha_t^{i(0)} = \alpha_0^{i(0)}$.

On the other hand, the continuity of the logarithm function \ln and the operation \oplus implies the equality $\lim_{t \rightarrow \infty} \ln \alpha_t^{i(0)} = \ln \alpha_0^{i(0)}$. Hence,

$$\lim_{t \rightarrow \infty} \left(\ln \alpha_t^{i(0)} - \bigoplus_{j(0)=1}^{n_0} \ln \alpha_t^{j(0)} \right) = \ln \alpha_0^{i(0)} - \bigoplus_{j(0)=1}^{n_0} \ln \alpha_0^{j(0)}.$$

Therefore, $\lim_{t \rightarrow \infty} z_I^P(\mu_t) = z_I^P(\mu_0)$, i. e. the map z_I^P is continuous. Theorem 3.1 is proved. \square

Corollary 3.2. *For an arbitrary metrizable compact space X the spaces $P(X)$ and $I(X)$ are homomorphic.*

Proof. As well-known that a metrizable compact space has a dense countable subset. Let M be a dense countable set in X . For each n let M_n be a n -point subset of M , $n = 1, 2, \dots$, such that $M_1 \subset \dots \subset M_n \subset M_{n+1} \subset \dots$, and $\bigcup_{n=1}^{\infty} M_n = M$. One can directly verified that $\bigcup_{n=1}^{\infty} P(M_n)$ and $\bigcup_{n=1}^{\infty} I(M_n)$ are dense in $P(X)$ and $I(X)$ respectively.

Let $z_\infty : \bigcup_{n=1}^{\infty} P(M_n) \rightarrow \bigcup_{n=1}^{\infty} I(M_n)$ be such a map that $z_\infty|_{P(M_n)} = z_I^P$ for each $n = 1, 2, \dots$. Then z_∞ is a homeomorphism and continued over all $P(X)$ uniquely. Let $z : P(X) \rightarrow I(X)$ be this continuation. It is clear z is a homeomorphism. Corollary 3.2 is proved. \square

4. FUNCTORS P AND I ARE NOT ISOMORPHIC

A subset L of the space $C(X)$ is called [21] a max-plus-linear subspace in $C(X)$, if:

- 1) $\lambda_X \in L$ for each $\lambda \in \mathbb{R}$;
- 2) $\lambda \odot \varphi \in L$ for each $\lambda \in \mathbb{R}$ and $\varphi \in L$;
- 3) $\varphi \oplus \psi \in L$ for each $\varphi, \psi \in L$.

Lemma 4.1 ([21] The max-plus variant of the Hahn-Banach theorem). *Let L be a max-plus-linear subspace in $C(X)$. Let $\mu : L \rightarrow \mathbb{R}$ be a functional satisfying the conditions of normality (I1), homogeneity (I2) and additivity (I3) (with $C(X)$ replaced by L). For an arbitrary $\varphi_0 \in C(X) \setminus L$ there exists an extension of the functional μ satisfying the conditions of normality, homogeneity and additivity on the minimal max-plus-linear subspace $L' \text{ containing } L \cup \{\varphi_0\}$.*

Consider the following subset in $C(X \times Y)$:

$$C_0 = \left\{ \bigoplus_{i=1}^n \varphi_i \odot \psi_i : \varphi_i \in C(X) \text{ and } \psi_i \in C(Y), i = 1, \dots, n; n \in \mathbb{N} \right\}.$$

It is obvious that C_0 is a max-plus-linear subspace in $C(X \times Y)$.

For every pair $(\mu, \nu) \in I(X) \times I(Y)$ we put

$$(\mu \tilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_i \odot \psi_i \right) = \bigoplus_{i=1}^n \mu(\varphi_i) \odot \nu(\psi_i).$$

Proposition 4.2. *The constructed functional $\mu \tilde{\otimes} \nu$ satisfies the conditions of normality, homogeneity and additivity on C_0 .*

Proof. Each $c \in \mathbb{R}$ can be represented as $c_{X \times Y} = a_X \odot b_Y$, where $a, b \in \mathbb{R}$ and $a + b = c$. Therefore, $(\mu \tilde{\otimes} \nu)(c_{X \times Y}) = (\mu \tilde{\otimes} \nu)(a_X \odot b_Y) = \mu(a) \odot \nu(b) = a \odot b = c$.

Let $\lambda \in \mathbb{R}$ and $\bigoplus_{i=1}^n \varphi_i \odot \psi_i \in C_0$. Then

$$\begin{aligned} (\mu \tilde{\otimes} \nu) \left(\lambda \odot \bigoplus_{i=1}^n \varphi_i \odot \psi_i \right) &= (\mu \tilde{\otimes} \nu) \left(\bigoplus_{i=1}^n (\lambda \odot \varphi_i) \odot \psi_i \right) = \\ &= \bigoplus_{i=1}^n \mu(\lambda \odot \varphi_i) \odot \nu(\psi_i) = \bigoplus_{i=1}^n \lambda \odot \mu(\varphi_i) \odot \nu(\psi_i) = \\ &= \lambda \odot \bigoplus_{i=1}^n \mu(\varphi_i) \odot \nu(\psi_i) = \lambda \odot (\mu \tilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_i \odot \psi_i \right). \end{aligned}$$

Finally, let $\bigoplus_{i=1}^n \varphi_{1i} \odot \psi_{1i} \in C_0$ and $\bigoplus_{j=1}^m \varphi_{2j} \odot \psi_{2j} \in C_0$. Then

$$\begin{aligned} (\mu \tilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_{1i} \odot \psi_{1i} \oplus \bigoplus_{j=1}^m \varphi_{2j} \odot \psi_{2j} \right) &= \\ &= (\mu \tilde{\otimes} \nu) \left(\bigoplus_{k=1}^{n+m} \varphi_{ki} \odot \psi_{ki} \right) = \bigoplus_{k=1}^{n+m} \mu(\varphi_{ki}) \odot \nu(\psi_{ki}) = \\ &= \bigoplus_{i=1}^n \mu(\varphi_{1i}) \odot \nu(\psi_{1i}) \oplus \bigoplus_{j=1}^m \mu(\varphi_{2j}) \odot \nu(\psi_{2j}) = \\ &= (\mu \tilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_{1i} \odot \psi_{1i} \right) \oplus (\mu \tilde{\otimes} \nu) \left(\bigoplus_{j=1}^m \varphi_{2j} \odot \psi_{2j} \right). \end{aligned}$$

Proposition 4.2 is proved. □

Since C_0 is a max-plus-linear subspace in $C(X \times Y)$, according to Lemma 1 for the idempotent probability measure $\mu \tilde{\otimes} \nu$ there exists its extension ξ all over $C(X \times Y)$ which satisfies the conditions of normality, homogeneity and additivity on $C(X \times Y)$.

So, we have proved the following max-plus variant of the Fubini theorem.

Theorem 4.3. *For every pair $(\mu, \nu) \in I(X) \times I(Y)$ there exists an idempotent probability measure $\xi \in I(X \times Y)$ such that $\xi(\varphi \odot \psi) = \mu(\varphi) \odot \nu(\psi)$, $\varphi \in C(X)$, $\psi \in C(Y)$.*

From the results of work [18] (see section 3) it follows that if $|X| \geq 2, |Y| \geq 2$, then $\mu \tilde{\otimes} \nu$ has uncountable many extensions on $C(X \times Y)$. Put

$$\mu \otimes \nu = \bigoplus \left\{ \xi \in I(X \times Y) : \xi|_{C_0} = \mu \tilde{\otimes} \nu \right\}.$$

Similarly to the traditional case, $\mu \otimes \nu$ we call as a “tensor” product of idempotent probability measure μ and ν .

Further, to distinguish the tensor products we will use symbols \otimes_I and \otimes_P for the idempotent and traditional cases, respectively.

Let us give the classical option of the Fubini theorem.

Theorem 4.4 ([4]). *For every pair $(\mu, \nu) \in P(X) \times P(Y)$ there exists a **unique** probability measure $\mu \otimes_P \nu \in P(X \times Y)$ such that $(\mu \otimes_P \nu)(\varphi \cdot \psi) = \mu(\varphi) \cdot \nu(\psi)$, $\varphi \in C(X)$, $\psi \in C(Y)$.*

Now we need some concepts from the category theory [4], [15].

Let $F_i: \mathfrak{C} \rightarrow \mathfrak{C}'$, $i = 1, 2$, be to functors from the category $\mathfrak{C} = (\mathfrak{D}, \mathfrak{M})$ to the category $\mathfrak{C}' = (\mathfrak{D}', \mathfrak{M}')$. A family of morphisms $\Phi = \{\varphi_X: F_1(X) \rightarrow F_2(X) | X \in \mathfrak{D}\} \subset \mathfrak{M}'$ is said to be a *natural transformation* of the functor F_1 to the functor F_2 , if for each morphism $f: X \rightarrow Y$ of the category \mathfrak{C} a diagram

$$\begin{array}{ccc} F_1(X) & \xrightarrow{F_1(f)} & F_1(Y) \\ \varphi_X \downarrow & & \varphi_Y \downarrow \\ F_2(X) & \xrightarrow{F_2(f)} & F_2(Y) \end{array}$$

is commutative, i. e.

$$F_2(f) \circ \varphi_X = \varphi_Y \circ F_1(f).$$

If, for every object X in \mathfrak{C} , the morphism f_X is an isomorphism in \mathfrak{C}' , then $\Phi = \{f_X\}$ is said to be a *natural isomorphism* (or sometimes *natural equivalence* or *isomorphism* of functors). Two functors $F_i: \mathfrak{C} \rightarrow \mathfrak{C}'$, $i = 1, 2$, are called *naturally isomorphic* or simply *isomorphic* if there exists a natural isomorphism from F_1 to F_2 .

Our goal is to show that the functors P and I are not isomorphic.

Example 4.5. Consider the sets $X = \{a, b, c\}$, $Y = \{a, b\}$, $Z = \{a, c\}$, where a, b, c are different points (these sets are supplied with discrete topologies). Define the following maps:

$$\begin{aligned} f: X &\longrightarrow Y, & f(a) = f(c) &= a, & f(b) &= b, \\ g: X &\longrightarrow Z, & g(a) = g(b) &= a, & g(c) &= c. \end{aligned}$$

Consider compact Hausdorff spaces $X, Y \times Z$, and the map $(f, g): X \longrightarrow Y \times Z$.

It suffices to show the map $(P(f), P(g))$ has a property that the map $(I(f), I(g))$ does not possess it.

At first we show the map $(P(f), P(g)): P(X) \longrightarrow P(Y) \times P(Z)$ is an embedding. In fact, for any pair of probability measures

$$\mu = \alpha_1\delta_a + \alpha_2\delta_b + \alpha_3\delta_c, \quad \nu = \beta_1\delta_a + \beta_2\delta_b + \beta_3\delta_c,$$

with positive $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \alpha_1 + \alpha_2 + \alpha_3 = 1, \beta_1 + \beta_2 + \beta_3 = 1$, the following equalities take place

$$\begin{aligned} P(f)(\mu) &= (\alpha_1 + \alpha_3)\delta_a + \alpha_2\delta_b, & P(f)(\nu) &= (\beta_1 + \beta_3)\delta_a + \beta_2\delta_b, \\ P(g)(\mu) &= (\alpha_1 + \alpha_2)\delta_a + \alpha_3\delta_c, & P(g)(\nu) &= (\beta_1 + \beta_2)\delta_a + \beta_3\delta_c. \end{aligned}$$

Therefore, $(P(f), P(g))(\mu) = (P(f), P(g))(\nu)$ if and only if

$$(1.P) \quad \begin{cases} \alpha_1 + \alpha_3 = \beta_1 + \beta_3, \\ \alpha_2 = \beta_2, \\ \alpha_1 + \alpha_2 = \beta_1 + \beta_2, \\ \alpha_3 = \beta_3. \end{cases}$$

System (1.P) has a unique solution $\alpha_1 = \beta_1, \alpha_2 = \beta_2$ and $\alpha_3 = \beta_3$. Hence, $\mu = \nu$. Thus, $(P(f), P(g))(\mu) = (P(f), P(g))(\nu)$ if and only if $\mu = \nu$, i. e. $(P(f), P(g)): P(X) \rightarrow P(Y) \times P(Z)$ is an embedding. Consequently, the diagram

$$\begin{array}{ccc} P(X) & \xrightarrow{(P(f), P(g))} & P(Y) \times P(Z) \\ P((f, g)) \downarrow & \swarrow \otimes_P & \\ P(Y \times Z) & & \end{array}$$

holds, i. e.

$$(2.P) \quad P((f, g)) = \otimes_P \circ (P(f), P(g)),$$

where the map $\otimes_P: P(Y) \times P(Z) \rightarrow P(Y \times Z)$ acts as $\otimes_P(\mu, \nu) = \mu \otimes_P \nu$.

Remind, the uniqueness of the solution of system (1.P) and equality (2.P) may be considered as corollaries of Theorem 4.4.

Now we show that the map $(I(f), I(g)): I(X) \rightarrow I(Y) \times I(Z)$ is not an embedding. Really, for idempotent probability measures

$$\mu = \lambda_1 \odot \delta_a \oplus \lambda_2 \odot \delta_b \oplus \lambda_3 \odot \delta_c, \quad \nu = \gamma_1 \odot \delta_a \oplus \gamma_2 \odot \delta_b \oplus \gamma_3 \odot \delta_c,$$

with $-\infty < \lambda_1, \lambda_2, \lambda_3, \gamma_1, \gamma_2, \gamma_3 \leq 0$ and $\lambda_1 \oplus \lambda_2 \oplus \lambda_3 = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 = 0$, the following equalities hold

$$\begin{aligned} I(f)(\mu) &= (\lambda_1 \oplus \lambda_3) \odot \delta_a \oplus \lambda_2 \odot \delta_b, & I(f)(\nu) &= (\gamma_1 \oplus \gamma_3) \odot \delta_a \oplus \gamma_2 \odot \delta_b, \\ I(g)(\mu) &= (\lambda_1 \oplus \lambda_2) \odot \delta_a \oplus \lambda_3 \odot \delta_c, & I(g)(\nu) &= (\gamma_1 \oplus \gamma_2) \odot \delta_a \oplus \gamma_3 \odot \delta_c. \end{aligned}$$

The equality $(I(f), I(g))(\mu) = (I(f), I(g))(\nu)$ is true if and only if

$$(1.I) \quad \begin{cases} \lambda_1 \oplus \lambda_3 = \gamma_1 \oplus \gamma_3, \\ \lambda_2 = \gamma_2, \\ \lambda_1 \oplus \lambda_2 = \gamma_1 \oplus \gamma_2, \\ \lambda_3 = \gamma_3. \end{cases}$$

System (1.I) has infinitely many solutions. For example, for every pair of λ_1 and γ_1 with $-\infty < \lambda_1 \leq 0$, $-\infty < \gamma_1 \leq 0$ a 6-tuple $(\lambda_1, \gamma_1, 0, 0, 0, 0)$ is its solution. The equality $(I(f), I(g))(\mu) = (I(f), I(g))(\nu)$ is true for this 6-tuple although $\lambda_1 \neq \gamma_1$. This means that the map $(I(f), I(g))$ is not an embedding. That means that the following diagram

$$\begin{array}{ccc} I(X) & \xrightarrow{(I(f), I(g))} & I(Y) \times I(Z) \\ I((f, g)) \downarrow & & \swarrow \otimes_I \\ & & I(Y \times Z) \end{array}$$

does not hold, i. e. the equality

$$(2.I) \quad I((f, g)) = \otimes_I \circ (I(f), I(g))$$

is wrong.

To present the existence of infinitely many solutions of system (1.I), or relation (2.I) consider idempotent probability measures

$$\mu = -1 \odot \delta_a \oplus 0 \odot \delta_b \oplus 0 \odot \delta_c, \quad \nu = -2 \odot \delta_a \oplus 0 \odot \delta_b \oplus 0 \odot \delta_c.$$

Then

$$\begin{aligned} I(f)(\mu) &= 0 \odot \delta_a \oplus 0 \odot \delta_b, & I(f)(\nu) &= 0 \odot \delta_a \oplus 0 \odot \delta_b, \\ I(g)(\mu) &= 0 \odot \delta_a \oplus 0 \odot \delta_c, & I(g)(\nu) &= 0 \odot \delta_a \oplus 0 \odot \delta_c, \end{aligned}$$

and

$$(I(f), I(g))(\mu) = (0 \odot \delta_a \oplus 0 \odot \delta_b, 0 \odot \delta_a \oplus 0 \odot \delta_c) = (I(f), I(g))(\nu),$$

which yields

$$I(f)(\mu) \otimes_I I(g)(\mu) = 0 \odot \delta_{(a,a)} \oplus 0 \odot \delta_{(b,c)} = I(f)(\nu) \otimes_I I(g)(\nu).$$

On the other hand since

$$(f, g)(a) = (a, a), \quad (f, g)(b) = (b, a), \quad (f, g)(c) = (a, c),$$

we have

$$\begin{aligned} I((f, g))(\mu) &= -1 \odot \delta_{(a, a)} \oplus 0 \odot \delta_{(b, a)} \oplus 0 \odot \delta_{(a, c)}, \\ I((f, g))(\nu) &= -2 \odot \delta_{(a, a)} \oplus 0 \odot \delta_{(b, a)} \oplus 0 \odot \delta_{(a, c)}. \end{aligned}$$

So,

$$I((f, g))(\mu) \neq I((f, g))(\nu).$$

Moreover,

$$I((f, g))(\mu) \neq \otimes_I \circ (I(f), I(g))(\mu)$$

and

$$I((f, g))(\nu) \neq \otimes_I \circ (I(f), I(g))(\nu).$$

Thus,

- (1) The map $(P(f), P(g)): P(X) \rightarrow P(Y) \times P(Z)$ is an embedding, while the map $(I(f), I(g)): I(X) \rightarrow I(Y) \times I(Z)$ is not an embedding.
- (2) For the embedding $(f, g): X \rightarrow Y \times Z$ the embedding $P((f, g)): PX \rightarrow P(Y \times Z)$ might be defined by the rule (2.P), while the embedding $I((f, g)): I(X) \rightarrow I(Y \times Z)$ does not have such a decomposition.
- (3) Regardless of the natural transformation $\Phi = \{\varphi_X: P(X) \rightarrow I(X) : X \in \mathbf{Comp}\}$ a diagram

$$\begin{array}{ccc} P(X) & \xrightarrow{(P(f), P(g))} & P(Y) \times P(Z) \\ \varphi_X \downarrow & & \downarrow \varphi_Y \times \varphi_Z \\ I(X) & \xrightarrow{(I(f), I(g))} & I(Y) \times I(Z) \end{array}$$

can not be commutative.

So, we came to the following important **Conclusion**: although for a metrizable Hausdorff compact X the spaces $I(X)$ and $P(X)$ are homeomorphic, Example 4.5 shows, the difference between the constructions P and I appears even on finite sets. Thus, the functors P and I are not isomorphic.

Note this conclusion was proclaimed in [21] as Proposition 2.15. But it was not equipped with detailed proof.

5. ON A METRICISE CRITERION OF THE COMPACT HAUSDORFF SPACES

The well-known M. Katětov's theorem states [9] that the hereditary normality of the cube X^3 of a Hausdorff compact space X follows metrizability of X . In 1989, V. V. Fedorchuk generalized [3] Katětov's theorem for a normal functors of the degree ≥ 3 , acting in the category \mathbf{Comp} . Many publications in the field of general topology are devoted to the issues of Katětov's theorem and problem.

The set of all nonempty closed subsets of the topological space X is denoted by $\exp X$. For open subsets $U_1, \dots, U_n \subset X$ a family of the sets of the type

$$O\langle U_1, \dots, U_n \rangle = \{F : F \in \exp X, F \subset \bigcup_{i=1}^n U_i, F \cap U_i \neq \emptyset, i = \dots, n\}$$

forms a base of a topology on $\exp X$. This topology is called the *Vietoris topology*, the set $\exp X$ equipped with the Vietoris topology is called a *hyperspace* of the topological space X .

For a compact X its hyperspace $\exp X$ is a compact. For the compact X , the natural number n , the functor F we put

$$F_n = \{a \in F(X) : |\text{supp } a| \leq n\},$$

$$F_n^0 = F_n(X) \setminus F_{n-1}(X),$$

where

$$\text{supp } a = \bigcap \{A \subset X : \overline{A} = A, a \in F(A)\}$$

is a support of the element $a \in F(X)$. In particular, $\exp_n X = \{K \in \exp X : |K| \leq n\}$, $\exp_n^0 = \exp_n X \setminus \exp_{n-1} X$, and $I_n = \{\mu \in I(X) : |\text{supp } \mu| \leq n\}$, $I_n^0 = I_n(X) \setminus I_{n-1}(X)$.

Let τ be an uncountable cardinal number. Put $N_\tau = \{x : x < \tau\}$. Provide N_τ with the discrete topology. Then it becomes a local compact Hausdorff space (which is not compact, since $|N_\tau| = \tau > \aleph_0$). By $\alpha N_\tau = N_\tau \cup \{p\}$ we denote its one-point compactification, where $p \notin N_\tau$.

In [3] (Proposition 1) it was shown that if τ is an uncountable cardinal then $\exp_2 \alpha N_\tau$ is not hereditary normal. We claim the following statement.

Proposition 5.1. *For every uncountable cardinal number τ the space $\exp_3^0 \alpha N_\tau$ is not normal.*

Proof. Obviously, there exist disjoint subsets F_1 and F_2 of N_τ such that F_1 is uncountable and F_2 is countable. Take a point $x_0 \in F_1 \cup F_2$. Choose subsets A_1 and A_2 of the space $\exp_3^0 \alpha N_\tau$, assuming

$$A_1 = \{\{p, x, x_0\} : x \in F_1 \setminus \{x_0\}\}, \quad A_2 = \{\{p, x', x_0\} : x' \in F_2 \setminus \{x_0\}\},$$

Obviously, $A_1 \cap A_2 = \emptyset$.

Let $F = \{x_1, x_2, x_3\} \in \exp_3^0 \alpha N_\tau \setminus A_1$. The set $O(\{x_1\}, \{x_2\}, \{x_3\})$ is an open neighbourhood of the set F which does not intersect A_1 . Hence, the set A_1 is closed in $\exp_3^0 \alpha N_\tau$. Similarly one can check that A_2 is closed in $\exp_3^0 \alpha N_\tau$.

For each $x \in N_\tau$ we put $U_x = O(\alpha N_\tau \setminus \{x_0, x\}, \{x_0\}, \{x\}) \cap \exp_3^0 \alpha N_\tau$. It is easy to see that the smallest by inclusion neighbourhoods of the sets A_1 and A_2 in $\exp_3^0 (\alpha N_\tau)$ are the sets $OA_1 = \bigcup_{x \in F_1} U_x$ and $OA_2 = \bigcup_{x \in F_2} U_x$, respectively. For the set $\{a, b, x_0\}$, where $a \in F_1, b \in F_2$, we have

$$\{a, b, x_0\} \in O(\alpha N_\tau \setminus \{x_0, a\}, \{x_0\}, \{a\}) \subset OA_1,$$

$$\{a, b, x_0\} \in O(\alpha N_\tau \setminus \{x_0, b\}, \{x_0\}, \{b\}) \subset OA_2.$$

This means, $OA_1 \cap OA_2 \neq \emptyset$. Proposition 5.1 is proved. \square

Proposition 5.2. *Let τ is an uncountable cardinal number. Then $I_3^0(\alpha N_\tau)$ is not normal space.*

Proof. For each compact X the set $\exp_3^0 X$ is closed in $I_3^0(X)$. Really, a correspondence $\{a, b, c\} \mapsto 0 \odot \delta_a \oplus 0 \odot \delta_b \oplus 0 \odot \delta_c$ establishes an identical embedding of $\exp_3^0 X$ into $I_3^0(X)$. This embedding is continuous and it is closed map. On the other hand the normality is a hereditary property for the closed subsets of the space. Therefore according to Proposition 5.1 the space $I_3^0(\alpha N_\tau)$ is not normal. Proposition 5.2 is proved. \square

In [3] (Theorem 2) it was established that if for a normal functor F of degree ≥ 3 the Hausdorff compact space $F(X)$ is hereditary normal then a Hausdorff compact space X is metrizable. We get modified shape of this result for the functor I , and it might be considered as a metricize criterion of compact Hausdorff spaces.

Theorem 5.3. *Let X be a compact Hausdorff space. If $I_3(X) \setminus X$ is a hereditarily normal space then X is metrizable.*

Proof. Suppose the compact X is non-metrizable. If X has a unique nonisolated point then X is homeomorphic to αN_τ for $\tau = |X| > \omega$. Proposition 5.2 implies that $I_3^0(X)$ is not normal. But according to the condition $I_3^0(X)$ must be normal as a subset of the hereditarily normal space $I_3(X) \setminus X$. We get a contradiction.

Now let a and b be distinguished nonisolated points of the compact X . There are open neighbourhoods U and V of points a and b , respectively, such that $\overline{U} \cap \overline{V} = \emptyset$.

We consider set $Z = \overline{U} \times \exp_2 \overline{V}$ and by the formula $\lambda(x, y, z) = 0 \odot \delta_x \oplus 0 \odot \delta_y \oplus 0 \odot \delta_z$ we define the topological embedding $\lambda : Z \rightarrow I_3(X) \setminus X$. The result of M. Katetov [9, Corollary 1] (which asserts the perfectly normality of the factor X under the condition of hereditary normality of the product of $X \times Y$) implies that the factor $\exp_2 \overline{V}$ of the product $Z = \overline{U} \times \exp_2 \overline{V}$ is perfectly normal. Further, applying the result of V. V. Fedorchuk [3] (which asserts the metrizable of the compact X if for a normal functor F of degree ≤ 2 the space $F(X)$ is perfectly normal) we conclude that \overline{V} is metrizable. Similarly, one can show that \overline{U} is metrizable. Therefore each nonisolated point of the compact X has a metrizable closed neighbourhood. Hence the compact X is locally metrizable. Therefore it is metrizable. Theorem 5.3 is proved. \square

Corollary 5.4. *Let X be a compact Hausdorff space and $n \geq 3$. If $I_n(X) \setminus X$ is a hereditarily normal space then X is metrizable.*

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