A Proof Using Böhme's Lemma That no Petersen Family Graph Has a Flat Embedding

J. Foisy, C. Jacobs, T. Paquin, M. Schalizki, and H. Stringer*

Abstract - Sachs and Conway-Gordon used linking number and a beautiful counting argument to prove that every graph in the Petersen family is intrinsically linked (have a pair of disjoint cycles that form a nonsplit link in every spatial embedding) and thus each family member has no flat spatial embedding (an embedding for which every cycle bounds a disk with interior disjoint from the graph). We give an alternate proof that every Petersen family graph has no flat embedding by applying Böhme's Lemma and the Jordan-Brouwer Separation Theorem.

Keywords : spatial graph; intrinsically linked; Petersen family

Mathematics Subject Classification (2020) **:** 57M15; 05C10

1 Introduction

Sachs [6], [7] and Conway-Gordon [3] (for K_6, the complete graph on 6 vertices) proved, using the linking number and a beautiful counting argument, that every graph in the Petersen family is intrinsically linked (that is, has a pair of disjoint cycles that form a nonsplit link in every spatial embedding). As a consequence of being intrinsically linked, every graph in the Petersen family has no flat embedding. A *flat* spatial embedding is one in which every cycle bounds a disk with interior disjoint from the graph. When such a disk is attached to a cycle, we say that cycle has been *paneled*.

We have devised a new family of proofs that every Petersen family graph has no flat embedding, without using intrinsic linking. We sketch our methods here. They were inspired by the proofs that K_5 and $K_{3,3}$ (the complete bipartite graph with two partitions each of size 3) are nonplanar using the Jordan Curve Theorem (see, for example, [4] Theorem 64.2 and Theorem 64.4).

Given a graph G with a flat embedding, consider a set of cycles in G such that, pair-wise, intersections are either connected or empty. We call such a set of cycles a *Böhme system*. Böhme's Lemma [2] concludes that for a Böhme system of cycles in a flat embedding of G, it is possible to panel the cycles in the system simultaneously–that is,

*This work was completed as part of a United States National Security Agency grant H89230-22-1-0008 Clarkson University: Summer (2022) Research Experience for Undergraduates in Mathematics

each cycle bounds a disk with interior disjoint from the graph and from all other disks. The Jordan-Brouwer Separation Theorem then allows us to conclude that any sphere resulting from paneling a Böhme system (which we will call a *Böhme sphere*) will separate space into an inside (bounded) and an outside (unbounded) component. Since we work in the PL category, by Alexander's result [1], the inside component is a topological ball.

For the sake of contradiction, we suppose a Petersen family graph has a flat embedding and then find an appropriate Böhme system. We then argue that some resulting Böhme sphere must be intersected by an edge of the embedded graph, which contradicts flatness. In this way, we prove that each Petersen family graph cannot have a flat embedding. In the next section, we provide further details for each graph in the Petersen Family.

We conclude this introductory section with some terminology and background. We work in the PL category throughout.

Suppose a graph G has a three cycle with vertices a, b, c. To perform a Δ -Y exchange on the three-cycle in G, an extra vertex y is added, edges ab, bc, and ca are removed, and the edges ay, by, and cy are added to transform G into G'. If a graph H has a degree three vertex, y, to perform a Y-Δ exchange on H at y, we begin with the Y subgraph, having vertices a, b, c, y and edges ay, by, and cy. We remove the y vertex and all edges incident to y and then add edges ab, bc, and ca to make a triangle and transform H to the graph H'.

The Petersen family of graphs consists of K_6 and $K_{3,3,1}$ and the graphs obtained from K_6 and $K_{3,3,1}$ by $\Delta - Y$ exchanges (equivalently, from K_6 by $\Delta - Y$ and $Y - \Delta$ exchanges). We will denote the remaining graphs in this family, with subscripts indicating the number of vertices, as P_7, P_8, P_9, P_{10} (which is the classic Petersen graph) and $K_{4,4} - e$. Robertson, Seymour and Thomas proved Sachs' conjecture that the Petersen family of graphs form the complete minor-minimal set of graphs that are intrinsically linked. They further showed that a graph has a flat embedding if and only if it has a linkless embedding [5].

2 Proofs

In this section, we go through details of the proof for each graph in the Petersen family.

2.1 K_6

An *induced subgraph* of a graph G is another graph, formed from a subset of the vertices of G and all of the edges of G connecting pairs of vertices in that subset.

Denote the vertices of K_6 as $\{a, b, c, d, e, f\}$. By way of contradiction suppose that $G = K_6$ has a flat embedding. Consider the induced subgraphs, $H_1 = G[\{b, c, e, f\}], H_2 = G[\{a, b, e, f\}]$, and $H_3 = G[\{b, d, e, f\}]$, where $G[\{a, b, c, d\}]$ is the subgraph of K_6 induced by the vertices a, b, c, d. Notice the cycle $(b, e, f) \in H_i$ for $i = 1, 2, 3$, see Figure 1.

Let S_i denote the collection of induced cycles in the induced subgraph H_i:

$$S_1 = \{(b, e, f), (b, c, e), (c, e, f), (b, c, f)\} \subseteq H_1$$
$$S_2 = \{(b, e, f), (a, b, e), (a, e, f), (a, b, f)\} \subseteq H_2$$
$$S_3 = \{(b, e, f), (b, d, e), (d, e, f), (b, d, f)\} \subseteq H_3$$

Let $\mathcal{S} = \cup_{i=1}^{3} S_i$.

Take a flat embedding of K_6. As all the cycles of $\mathcal{S}$ have connected pairwise intersections, we can apply Böhme's Lemma to get simultaneous paneling disks of the ten 3-cycles that make up $\mathcal{S}$. As we panel each S_i, we form a sphere T_i, then by Jordan-Brouwer Theorem applied to, say, T_1, there are inside and outside components, B_1 and B_1' respectively, with $B_1 \cup B_1' = \mathbb{R}^3 - T_1$, and, say, B_1 a ball (which follows from Alexander's result [1] in the PL category). Without loss of generality as the edge ad is disjoint from T_1, we may assume that $ad \subset B_1$. As $ad \subset B_1$ and $a \in T_2$, $d \in T_3$ then $T_2, T_3 \subset B_1 \cup T_1$. This means that T_2 and T_3 are in the closure of B_1. Now similarly observe there exists, via Jordan-Brouwer, an inside and outside region for T_2. Without loss of generality suppose that T_3 is in the closure of the inner region (ball) of T_2. As $d \notin T_2$ and $d \in T_3$, d is in the inner ball of T_2. As T_2 is inside the closure of B_1 and $c \in T_1$, c is outside T_2. As edge cd exists and as c and d are on opposite sides of T_2, edge cd must intersect a disk of T_2. This brings us a contradiction. Thus K_6 has no flat embedding.

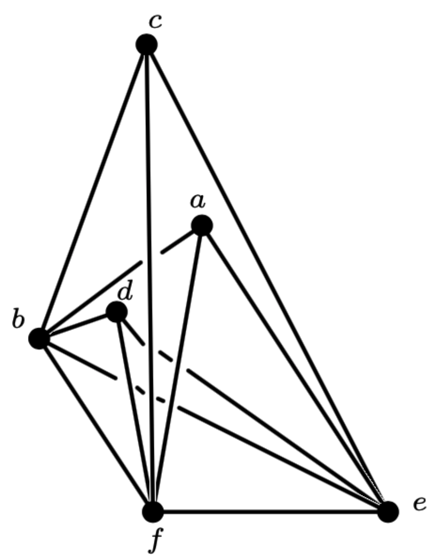

Figure 1: A spatial embedding of the subgraph of K_6 formed by $\mathcal{H}$.

The remaining cases

For all remaining Petersen family graphs except P_{10} and $K_{4,4}-e$, similarly to the proof for K_6, we consider a flat embedding, and a cycle C with vertex set $V(C)$, together with three vertices disjoint from $V(C)$; v_1, v_2 and v_3, such that each $G[V(C) \cup \{v_i\}]$ forms a Böhme system. After paneling each Böhme system, the result is three spheres that intersect only along C and the disk that panels C. In each case, two vertices are on opposite sides of a resulting Böhme sphere. In each case, the two vertices are connected either by an edge, or path consisting of two edges that is disjoint from $G[V(C) \cup \{v_i\}]$ for the appropriate i, and the result is a contradiction of flatness. More details follow.

2.2 P_7

Denote the six vertices of K_6 by $\{a, b, c, 1, 2, 3\}$, and suppose that a $\Delta - Y$ exchange took place between $1, 2$, and 3, with new vertex y. By way of contradiction, suppose that $G = P_7$ has a flat embedding. Consider the induced subgraphs:

$$H_1 = G[\{a, b, c, 1\}]$$
$$H_2 = G[\{a, b, c, 2\}]$$
$$H_3 = G[\{a, b, c, 3\}]$$

Note that each H_i contains the cycle (a, b, c). One can check that the induced cycles in $H_1 \cup H_2 \cup H_3 = K_{3,1,1,1}$ form a Böhme system. Take a flat embedding of P_7. The simultaneous paneling of the Böhme system turns each H_i into a Böhme sphere. Note that since the H_i share exactly one cycle, for a paneled flat embedding, the determined Böhme spheres bound balls (let B_i be the ball associated to H_i). We can conclude that each pair of balls is comparable, set-wise. In particular, once one Böhme sphere (and associated ball) is in place, the next has to go all to one side or the other, by flatness, the definition of Böhme system, and the Jordan-Brouwer Theorem. Without loss of generality, we may assume $B_1 \subseteq B_2 \subseteq B_3$. Then the path consisting of two edges connecting 1 and 3 through y must intersect a panel of H_2, a contradiction of flatness. This is analogous to the argument for K_6, except the vertices $\{1, 2, 3\}$ are connected by a Y subgraph instead of a triangle.

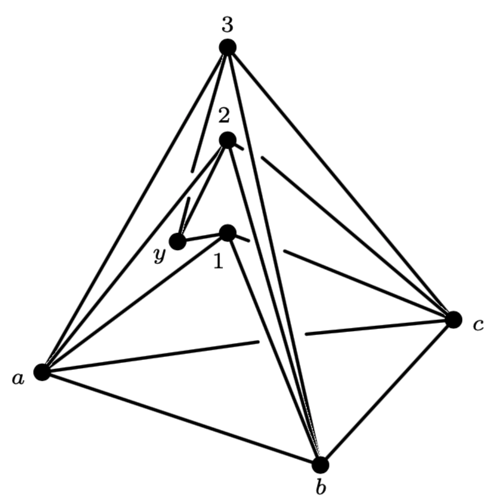

Figure 2: A spatial embedding of P_7.

2.3 $K_{4,4} - e$

Denote the vertices of $G = K_{4,4} - e$ as $V_1 = \{a, b, c, d\}$ and $V_2 = \{1, 2, 3, 4\}$ where all possible edges from V_1 to V_2 are present, except $e = 4a$

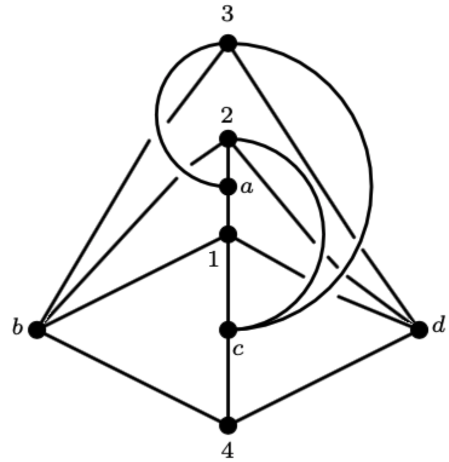

Figure 3: A spatial embedding of $K_{4,4} - e$. Possibly the edge $a3$ intersects a panel.

Assume by way of contradiction that $K_{4,4} - e$ has a flat embedding.

Consider the set of cycles of the induced subgraphs H_i listed below, together with their cycles, where $i \in \{1, 2, 3\}$.

$$H_1 = G[b, c, d, 1, 4]$$
$$H_2 = G[b, c, d, 3, 4]$$
$$H_3 = G[b, c, d, 3, 4]$$

Each has corresponding induced cycles: $S_1 = \{(1, b, 4, c), (1, c, 4, d), (1, b, 4, d)\} \subseteq H_1$

$$S_2 = \{(2, b, 4, c), (2, c, 4, d), (2, b, 4, d)\} \subseteq H_2$$
$$S_3 = \{(3, b, 4, c), (3, c, 4, d), (3, b, 4, d)\} \subseteq H_2$$

Notice that $S_1 \cup S_2 \cup S_3$, satisfies the hypotheses of Böhme's Lemma with edges forming $K_{4,3}$. By Böhme's lemma, we can simultaneously panel each of these cycles. Note that all paneled cycles pass through the vertex 4. By abuse of notation, denote the sphere resulting from paneling S_i also as S_i, where $i \in \{1, 2, 3\}$. Note that each H_i contains the Y on vertices $4, b, c, d$, as well as the vertex i. From here, the proof follows similarly to the argument used for P_7 (K_6), with the base cycle (triangle) C replaced with the Y, where for $i \neq j$, $S_i \cap S_j$ is precisely the Y.

2.4 $K_{3,3,1}$

Label the vertices of $G = K_{3,3,1}$ as in Figure 4. Assume by way of contradiction that $K_{3,3,1}$ has a flat embedding. Consider the collection of induced subgraphs:

$$H_1 = G[a, c, 1, 2, 3]$$
$$H_2 = G[a, b, c, 2, 3]$$
$$H_3 == G[a, c, v, 2, 3]$$

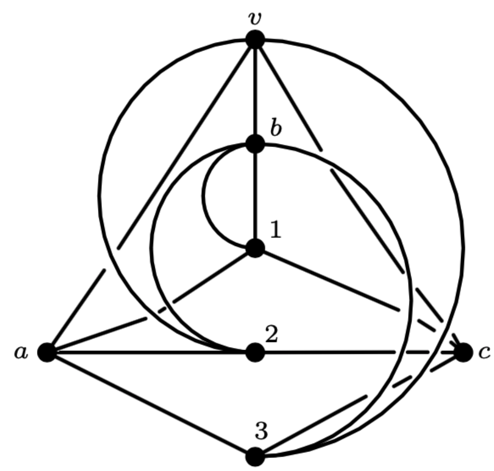

Figure 4: A spatial embedding of $K_{3,3,1}$.

Each has corresponding induced cycles:

$$S_1 = \{(a,2,c,3),(a,1,c,2),(a,1,c,3)\} \subseteq H_1$$
$$S_2 = \{(a,2,c,3),(3,b,2,c),(3,b,2,a)\} \subseteq H_2$$
$$S_3 = \{(a,2,c,3),(c,2,v),(c,v,3),(3,v,a),(a,v,2)\} \subseteq H_3$$

Notice that $S_1 \cup S_2 \cup S_3$ satisfies Böhme's lemma, and after paneling each, S_i forms a Böhme sphere. These Böhme spheres share precisely the same base cycle $(a,2,c,3)$ with the corresponding paneling disk. From here, the argument follows similarly to the argument for K_6.

2.5 P_8

Denote the vertices of P_8 by $\{v,a,b,c,1,2,3,y\}$, where $\{v\}$, $\{a,b,c\}$, and $\{1,2,3\}$ are the partitions in $K_{3,3,1}$, and triangle $(v,a,1)$ was exchanged to a Y subgraph, with resulting new Y (degree 3) vertex y in the resulting P_8.

By way of contradiction, suppose that $G = P_8$ has a flat embedding. Consider the induced subgraphs:

$$H_1 = G[v,a,b,1,2,y]$$
$$H_2 = G[a,b,c,1,2,y]$$
$$H_3 = G[a,b,1,2,3,y]$$

and their respective Böhme spheres formed by paneling their induced cycles, which is possible by Böhme's Lemma:

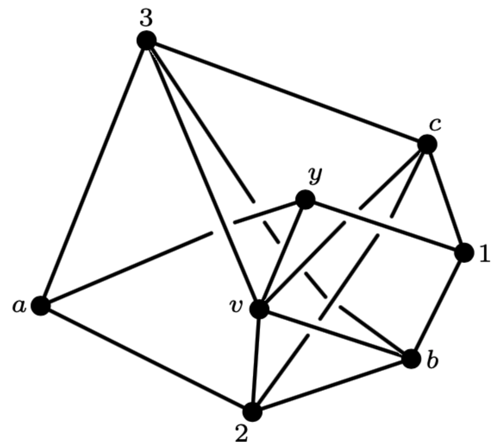

Figure 5: A spatial embedding of P_8.

$$S_1 = \{(a, y, 1, b, 2), (v, 2, a, y), (v, y, 1, b), (v, b, 2)\}$$
$$S_2 = \{(a, y, 1, b, 2), (a, y, 1, c, 2), (b, 1, c, 2)\}$$
$$S_3 = \{(a, y, 1, b, 2), (a, y, 1, b, 3), (a, 2, b, 3)\}$$

Note that each S_i contains the cycle $(a, y, 1, b, 2)$. From here, the argument follows analogously to the argument for K_6.

2.6 P_9

Let P_9 be labelled as in Figure 6.

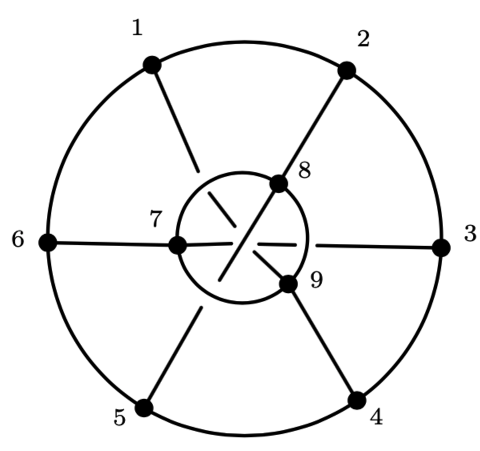

Figure 6: A spatial embedding of P_9.

Assume by way of contradiction that $G = P_9$ is has a flat embedding. Consider a flat embedding, and consider the sets of induced subgraphs:

$$H_1 = G[1,2,3,4,5,6,7]$$
$$H_2 = G[1,2,3,4,5,6,8]$$
$$H_3 = G[1,2,3,4,5,6,9]$$

Each induced subgraphs contains the following induced cycles:

$$S_1 = \{(1,2,3,4,5,6),(6,7,3,4,5),(6,7,3,2,1)\} \subseteq H_1$$
$$S_2 = \{(1,2,3,4,5,6),(2,8,5,4,3),(2,8,5,6,1)\} \subseteq H_2$$
$$S_3 = \{(1,2,3,4,5,6),(1,9,4,3,2),(1,9,4,5,6)\} \subseteq H_3$$

Note that each H_i contains the cycle $C = (1,2,3,4,5,6)$ and notice that by Böhme's lemma each of the cycles in $S_1 \cup S_2 \cup S_3$ can be simultaneously paneled so each S_i determines a sphere for $i = 1,2,3$ with, for $i \neq j$, the sphere determined by S_i intersecting the sphere determined by S_j only along C and its corresponding paneling disk. From here, the proof follows analogously to the proof for K_6.

2.7 P_{10}

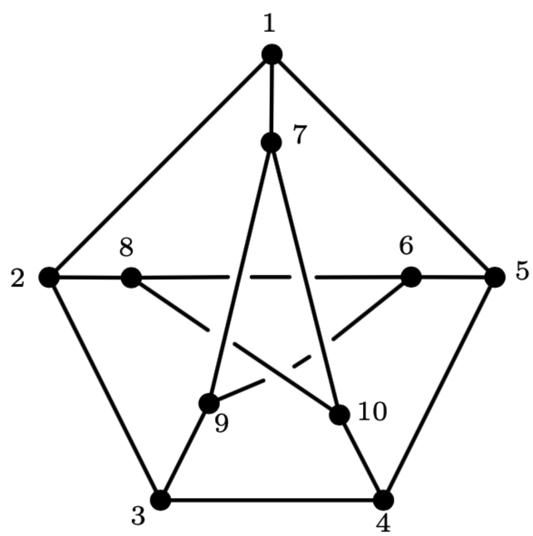

Figure 7: A spatial embedding of the Petersen graph.

Suppose by way of contradiction that $G = P_{10}$ has a flat embedding. Consider the induced subgraphs formed by each set of vertices below, where labels are from Figure 7.

$$H_1 = G[\{1,2,3,4,5,6,9\}]$$
$$H_2 = G[\{1,2,3,4,5,7,10\}]$$
$$H_3 = G[\{1,2,3,4,5,6,8\}]$$

Consider a flat embedding of P_{10}. Then by Böhme's lemma, each 5-cycle and 6-cycle in H_1, H_2, and H_3 can simultaneously bound disks to form three spheres, corresponding to each H_i. Note that the cycle $C = (1, 2, 3, 4, 5)$ is in each H_i.

By the Jordan-Brouwer Theorem, since vertex 7 is connected to vertex 10 by an edge, they must both be outside or both be inside of the sphere formed by H_1. Without loss of generality, assume both vertices are outside. Then the sphere created by H_2 is in the closure of the region outside of the sphere created by H_1. Since vertex 8 is connected to both 6 and 10 by an edge, vertex 8 must be between the sphere created by H_2 and the sphere created by H_1. Thus the sphere formed by H_3 is sandwiched between the sphere created by H_2 and the sphere created by H_1 (that is, it lies in the closure of the outer region formed by H_2 and in the closure of the inner region formed by H_1). But now since 7 is connected to 9 by an edge, and 7 is in H_2 and 9 is in H_1, the sphere formed by H_3 intersects the edge connecting 7 to 9. Hence a paneled disk must have been punctured by edge 79, contradicting flatness. In Figure 7, the paneled cycle $(2, 3, 4, 10, 8)$ from H_3 would be punctured. Therefore P_{10} has no flat embedding.

We note that the P_{10} case was similar to the earlier cases, except we needed a shared cycle C and 5 additional vertices, and H_1 and H_3 shared the vertex 6, in addition to the cycle C. The spheres they formed shared the base cycle (and disk) as well as the edge $5, 6$. Instead of having a triangle or Y connecting the extra vertices, here we have edges $8, 10$ and $7, 9$. Since 6 lies in both H_1 and H_3 but not in H_2; H_1 and H_3 must be on the same side of H_2. Even with a swap of positions of H_1 and H_3 from our original proof, the edge $8, 10$ would have intersected the Böhme sphere formed by H_1.

We could have alternatively done a $\Delta - Y$ exchange on P_9, on triangle $(7, 8, 9)$, to get P_{10} with the same Böhme system we used on P_9 and the proof details analogous to those used on P_9.

3 Concluding Note

As the proofs we presented were inspired by proofs of nonplanarity, we wonder if there are other analogs of nonplanarity that can be brought to light via Böhme's lemma.

Acknowledgments

We would like the referee for making helpful suggestions.

References

[1] J.W. Alexander, On the subdivision of 3-space by a polyhedron, *Proc. Natl. Acad. Sci. USA*, **10** (1924), 6–8.

[2] T. Böhme, On spatial representations of graphs, *Contemporary Methods in Graph Theory* (R. Bodendieck, Ed.), Bibliographisches Inst., Mannheim, (1990), 151–167.

[3] J.H. Conway, C.McA. Gordon, Knots and links in spatial graphs, *J. Graph Theory*, **7** (1983), 445–453.

[4] J.R. Munkres, *Topology*, Prentice Hall, Inc., Upper Saddle River, NJ, 2000.

[5] N. Robertson, P.D. Seymour, R. Thomas, Linkless Embeddings of Graphs in 3-Space, *Bull. Amer. Math. Soc (N.S.)*, **28** (1993), 84–89.

[6] H. Sachs, On a spatial analogue of Kuratowski's theorem on planar graphs—an open problem, *Graph theory (Lagow, 1981)*, Lecture Notes in Math. **1018**, Springer, Berlin, (1983), 230–241.

[7] H. Sachs, On spatial representations of finite graphs, *Colloq. Math. Soc. János Bolyai*, **37** (1984), 649–662.

Joel Foisy
Department of Mathematics
SUNY Potsdam
44 Pierrepont Ave.
Potsdam, NY 13676
E-mail: `foisyjs@potsdam.edu`

Catherine Jacobs
linkedin.com/in/catherine-jacobs-5aa337182
E-mail: `cj101@wellesley.edu`

Trinity Paquin
E-mail: `paquintrinity@gmail.com`

Morgan Schalizki
Department/School of Mathematical and Statistical Sciences
Clemson University
Clemson, SC 29634
E-mail: `mschali@clemson.edu`

Henry Stringer
E-mail: `henriotto@gmail.com`

Received: March 7, 2023 **Accepted**: November 11, 2024
Communicated by Darren A. Narayan