

Original Paper

An Analysis of Zeno's Paradox and Non-measurable Sets Based on Dialectical Infinity

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Received: November 7, 2021 Accepted: November 23, 2021 Online Published: December 11, 2021

doi:10.22158/jrph.v4n4p30

URL: <http://dx.doi.org/10.22158/jrph.v4n4p30>

Abstract

The Problem of Continuity and Discreteness is the basic problem of philosophy and mathematics. For a long time, there is no clear understanding of this problem, which leads to the stagnation of the problem, because the essence of the problem is a problem of finity and infinity. The essence of the philosophical thought on which the mathematical definition of "line segment is composed of dots" is the idea of actual infinity, and geometric dot is equivalent to algebraic zero in terms of measure properties. In view of the above contradictions, this paper presents two solutions satisfying both the philosophical and mathematical circles based on the view of dialectical infinity, and the authors make a deep analysis of Zeno's paradox and the non-measurable set based on both solutions.

Keywords

continuity, dot (point), actual infinity, the view of dialectical infinity, Zeno's Paradox, non-measurable set

The contradiction between continuity and discreteness, like the contradiction of finite and infinity, is the basic problem of mathematics and philosophy. However, for a long time, there is no scientific understanding of this problem, and the understanding of continuity and Zeno's Paradox is vague and ambiguous. Even in the definition of basic mathematical concepts, there are internal irreconcilable contradictions, such as *a line segment is composed of dots*, a dot has no measure, while a line segment has measure; these dots without measure constitute a line segment with measure, which is obviously contradictory and absurd. It is precisely because there is this inherent contradiction in the foundation of mathematics, which leads to many strange and absurd results in mathematics, which affects the image of mathematical certainty.

1. Analysis of the Problem of “Line Segment is Composed of Dots” by Dialectical Infinity View

1.1 Analysis of the Problem

On the mathematical definition of point (or dot), it is considered that a point has no measure and a line segment has measure, but on the other hand, mathematics stipulates that ‘a line segment is composed of points’, which leads to an inherent irreconcilable contradiction in mathematics. The *point* of mathematics is an absolute divided thing (which cannot be further divided). The line segment is composed of infinite points, which is composed of many points of *actual infinity*, thus a definition of *actual infinite property*. This idea of *actual infinity* inevitably leads to the lack of qualitative unity between line segment and point (*the essence of actual infinity is to separate the infinite process from the result qualitatively*), leading to the internal inconsistency between the point and the line segment, which leads to the irreconcilable contradiction, so that *the line segment is composed of points* is contradictory and absurd in philosophy.

Point, as a purely mathematical abstraction, is zero in length (measure), is nothingness and an imaginary thing. **This kind of geometric pure point, like the pure zero in algebra, is an absolute nothingness (emptiness), which has no inherent connection with space and continuity, and has no qualitative unity;** therefore, more points can only constitute a point of nothingness, can not constitute *continuity*. *Line segment* and *point* (or set of points) are two completely different concepts, no one can represent the other. There can be countless points of *nothingness* on the line segment, but what does this have to do with the line segment? Therefore, the line segment is not composed of points, and the face cannot be composed of straight lines and the space cannot be composed of faces.

However, if the *point* is regarded as an *impure* abstract, if it has a length, it represents a process rather than a *stationary object*, it is a variable; it can be seen as a limit process of shrinking a line segment, thus a line segment with *infinite small* measure, then *point* and *line segment* will have qualitative unity, and the line segment will become understandable by the composition of points.

Quality is the internal stipulation of things, and quantity is the external stipulation of things. **Dialectical Infinity View** (Zhang & Zhuang, 2019) holds that *real infinity* (or true infinity) is the connection and unity of the inner quality of *bad infinity*, so *real infinity* represents and reflects the quality of *bad infinity*; infinity is the unity of quality and quantity, bad (or evil) infinity represents quantity of infinity (motion), real infinity represents quality (law, commonness or connection) of infinity. The dialectical infinity view holds that the *actual infinity* separates the finite from the infinite, and looks at things from the point of view of stillness rather than movement, which has an internal irreconcilable contradiction, and is a metaphysical idealistic infinite view. As a mathematical abstract concept, the inherent quality of a line segment is continuity and if it is an infinite (*composed of infinite points*), because length is not the internal quality of *point*, therefore, the quality of line segment cannot be reflected in the quality of point. Therefore, to reflect the qualitative unity of the two, it must be that the *point* should also have a measure. On the other hand, only when the *point* has a measure, can it reflect the continuity of the line segment, and the line segment can be divided into parts that can be

divided forever, that is, divided into measured *points*, which is a process of *bad infinity* and inexhaustible. Therefore, we must give up the expression and definition of ‘*line segment is composed of points*’, which is an inherent contradiction, and give up the definition of actual infinite property.

1.2 Solutions

Looking at the analysis of the major philosophers and mathematicians in history (Zhang & Zhou, 2021), two solutions have surfaced. For the relationship between points and line segments, we have two acceptable understandings, both of which can solve the problem of continuity and discreteness well.

One is that “*a line segment is composed of points*”, but it must be recognized that *point* has a measure of *infinitesimal* (variable), that is, a point has a measure that is both zero and not zero, and the *point* is still essentially a line segment. The basis of this solution is to confirm the qualitative unity of point and line segment. Therefore, the *point* is still an infinite process rather than an end point, that is, *this point* is not the *point* in which the measure in mathematics is purely zero. In this way, the *line segment consists of points* becomes an infinite process, an endless process. This accords with the understanding of dialectical infinity, and there is no inherent contradiction. Hegel and Peirce proposed such a solution. This solution shows that the measure (quantity) of a measured point is a unity of being and not being, a changing infinitely small quantity, and a variable. Macroscopically, the measure of single point is zero, while relative microscopic, the measure of a single point is not zero, but an infinite small quantity (variable), which embodies an infinite process. Therefore, the measure of an infinite number of such points, macroscopically, may be zero or not, which requires a clear qualitative relationship between the *infinity* and the segment.

Another kind of understanding is to oppose “*a line segment consists of points*”, that is, the line segment is not composed of points, the line segment itself is a basic mathematical abstract object with measure (that is, continuity as a basic abstract object), the line segment has nothing to do with *points*, and has nothing to do with infinite and infinite processes. A point is another mathematical abstract object independent of a line segment. A point does not belong to a line segment. A point is imposed on a line segment. A point is simply a descriptive tool or marker for measuring size, location. From this point of view, there is no logical and philosophical basis to replace the *ergodicity* of infinite points with the *completion* of a finite long line segment. Kant has proposed a similar solution. This philosophical thought of denying *a line segment composed of points* is embodied in the ideas of Aristotle, Kant, Wittgenstein and other masters. This concept extends to the understanding of *number*, that is, the set of real numbers is a mark set, the number is the mark of the quantity, and the number corresponds to the line segment on the number axis. As Mr.WangHao said, “The measurement of length and volume is a combination of arithmetic and geometry, applying various units to calculate a number. As with the solution of the equation, this is a natural way to produce fractions and irrational numbers. The concept of *real numbers* arises from measurements that require absolute precision and, more precisely, can be infinitely improved.”

1.3 The Number Itself Does Not Reflect Continuity

Modern mathematics equates the set of real numbers with the continuum (line segment), so that each real number corresponds to a *mathematical point*, and the continuity of real numbers is used to reflect the continuity of the line segment. However, the number itself, like the point, cannot represent and reflect continuity, the essence of the number is representing separation.

In Hegel's view, the number itself cannot explain the continuity, the essence of the number reflects the separation; Hegel believes that quantity includes two links: continuity and dispersion, which is unified by quantity, that is, continuity and dispersion are dialectical unity through quantity. The essence of *quantity* originates from measurement, which is the calculation of length naturally, but not continuity itself. Number is the embodiment of quantity, so it is called quantity; **Hegel believes that “number is the absolute determinateness of quantity” (Hegel, 1969, p. 212), Engels believed that “number is the purest quantitative determination that we know” (Engels, p. 262).** Therefore, numbers can represent both discrete quantities, such as numbers, counts, and continuous quantities, such as units, scales (lengths). When the number represents a continuous quantity, it is based on the unit scale (a continuous) and the number is the measurement of the continuous system. Therefore, the number hides the continuity through the *unit* and discretizes the continuity object, so the number itself cannot explain the continuity, and the essence of quantity is to embody the separation.

Hegel emphasizes the unity of continuity and separation (discreteness). He points out that, “Conversely, in discrete magnitude continuity is not to be overlooked; this moment is, as has been shown, the one as unity.” (Hegel, 1969, p. 200); That is, the basis of number is unit, scale, so the fundamental characteristic of number is measurement. An essence of unit, scale is a quantum, is “a quantum, to which a determinate being or a quality is attached.” (Hegel, 2015, p. 172), is the unity of quality and quantity. Hegel believes that the essence of quantity is separation, which does not directly reflect continuity. He states: “*Immediate* quantity is continuous magnitude. But quantity is not an immediate at all; immediacy is a determinateness the sublatedness of which is quantity itself. It is, therefore, to be posited in the determinateness immanent in it, and this is the one. Quantity is *discrete magnitude*.” (Hegel, 1969, p. 200)

Hegel introduced the concept of *number* in detail, and pointed out that the number from the separation link is the sum, from the continuous link is the unit. Number embodies *many*, which can represent both continuous and discrete quantities. *One* as the embodiment of *unit*, sublation of continuity, that is, *one* does not reflect continuity and *unit* itself reflects continuity; The number is based on *unit* (one), and the number has continuity in *unit*, which deeply reveals that the number hides continuity through *unit*. Hegel states, “Quantum, which to begin with is quantity with a determinateness or limit in general is, in its complete determinateness, number.” (Hegel, 1969, p. 202), “Number exhibits the development and perfect character of the Quantum. Like the One, the medium in which it exists, Number involves two qualitative factors or functions; Annumeration or Sum, which depends on the discrete influence, and Unity, which depends on continuity.” (Hegel, 2015, p. 163)

Therefore, the number hides the continuity, the number reflects the continuity by the *unit*, and the number itself cannot explain continuity. Hegel states, “Quantum completely posited in these determinations is *number*. The complete positedness lies in the existence of the limit as a *plurality* and so in its distinction from the unity. **Consequently, number appears as a discrete magnitude, but in the unity it equally possesses continuity.** It is, therefore, also quantum in its complete *determinateness*, for in it the limit is present as a specific *plurality* which has for its principle the one, the absolutely determinate. Continuity, in which the one is present only *in principle*, as a sublated moment—posited as a unity—is the form of indeterminateness.” (Hegel, 1969, p. 203)

2. Analysis of Zeno’s Paradox

2.1 Regarding the Dichotomy, It Is Considered That the Motion Does not Exist. The Content Is: The Object must Reach Half of the Whole Journey before Reaching Its Destination, and This Requirement Can Be Carried out Indefinitely, so It Can never Reach the End.

The first explanation of Dialectical Infinity (that is, a point is essentially a line segment): because a *point* is always a line segment, this distance is not made up of infinite *points*. In a limited time, we can cross a distance or a finite number of *point segments*. Or, according to Aristotle, the structure of time and space is exactly the same, then, it is entirely possible to cross a finite “*point of space*” (that is, a distance) within a finite “*point of time*” (that is, a period of time), or to cross an infinite (potential infinite) *point of space* in an infinite (that is, potential infinite) *point of time*. Aristotle wrote in *Physics*, “For example, if time is infinite in the extension of two directions, the quantity is infinite in the extension of two directions; if time is infinite in division, the quantity is infinite in division; if time is infinite in extension and division, then the quantity is infinite in both aspects.” (Aristotle, p. 168), “Therefore, we can neither pass through infinite quantity in finite time nor through finite quantity in infinite time. But, time is infinite, quantity is infinite, and quantity is infinite, time is infinite.” (Aristotle, p. 169)

The second explanation of Dialectical Infinity (that is, the line segment is independent of the point, the point is a mathematical point): Since the line segment is not made up of points, there is no internal correlation between the line segment and *the points of infinite many*, so we have nothing to do with ‘*an infinite or infinite process*’ to cross a distance in a limited time, just as we have stepped a distance, we cannot say that we have crossed an infinite.

2.2 Regarding the Immobility of Flying Arrow, That Is, a Flying Arrow Is Still.

The first explanation of Dialectical Infinity (that is, a point is essentially a line segment): The point is essentially a line segment, so the moment of *now* is still essentially a period of time, that is, time is composed of the time period of *now*. So the flying arrow is still flying at any time (that is, for a period of time).

If the time is regarded as an absolute moment with no length of time, that is, the line segment is consists of points with no measure, then we must get the contradictory conclusion mentioned above,

because the motion is gone there (the essence of the *actual infinity* is a denial of the motion), so the arrow flying is still. As Aristotle put it, “this conclusion is caused by treating time as being synthesized by *presents*.” (Aristotle, p. 192); He pointed out that “nothing can move in the *present*” (Aristotle, p. 172), “the movement of moving things and the stillness of still things must be for a while.” (Aristotle, p. 173) Time is reflected by movement. Only when we feel the change of movement can we feel time. Movement must be a movement in time.

The second explanation of Dialectical Infinity (that is, the line segment is independent of the point, time is independent of moment): That is, time has nothing to do with the point of *present* (moment); without the moment of *present*, the problem itself does not exist.

3. Analysis of non Lebesgue-measurable Sets

We begin by constructing a non-measurable set. For the rational number set Q , the following *family of sets* is defined: $x + Q$, **thereinto** $x \in R$, and R represents the set of real numbers. The *family of sets* comes from Q translation arbitrarily to left or to right (called as the *coset* of Q), each of which is equal to Q and is dense in R . These *cosets* have the following properties: One is that different *cosets* do not intersect each other, i.e., **if** $x - y \notin Q$, **then** $(x + Q) \cap (y + Q) = \emptyset$, **thereinto** $x \in R$, $y \in R$. The second is that the union of all the *cosets* is exactly R , i.e.,

$$\bigcup_{x \in R} (x + Q) = R$$

Below, we will specifically construct the non-measurable set. Since each *coset* $x + Q$ is dense in R , therefore $X = (x + Q) \cap [0, 1] \neq \emptyset$. One element is now selected from each of all these disjoint intersection X (from the Axiom of Choice), and these elements form a new set V which is a non-measurable set. Generally speaking, we decompose interval $[0,1]$ into disjoint countable sets of infinitely many (uncountable), and the measure of X is equal to 0, i.e., $\mu(X) = 0$, because the measure of a countable set is 0. Select one element from each of these disjoint sets X and form a new set V , which is a non-measurable set that has the same cardinal number as the interval $[0,1]$ and is an uncountable set.

Analysis of the measure of V . On the one hand, if $\mu(V) = 0$ and the interval $[0,1]$ is composed of such sets V of countable many (proof omitted), this will cause the measure of $[0,1]$ to be also 0. If $\mu(V) > 0$, on the other hand, this will cause the measure of the union of these sets V (i.e., interval $[0,1]$) to become infinite, which is obviously not possible. Therefore, V is a non-measurable set.

The fundamental reason why V becomes a non-measurable set lies in the basic mathematical contradiction that “a line segment is composed of points”. The existence of a non-measurable set deeply reflects and reveals the internal contradictions of basic mathematics.

The first explanation of Dialectical Infinity (that is, a point is essentially a line segment): In this case, a single *dot* has an *infinitesimal* measure (variable), namely, the *dot* has a measure that is both zero and not zero. Similarly, the measure of countable set X is an *infinitesimal* measure (variable). The

interval $[0,1]$ consists of countable infinitely many V sets, and each V set is mutually equivalent, so, the measure of V is also an *infinitesimals*, both zero and not zero; countable infinitely many *infinitesimals* constitute measure 1. It is only that the *infinitesimals* here is different from the *infinitesimals* of the measure of set X , and the relationship between them is determined by the intrinsic connection between their *properties of infinity*.

Hegel holds that infinity are comparable and can be compared only in the sense of *real infinity*, that is, in the sense of the qualitative determinateness and qualitative connectivity of infinity. The two sets of V and X (namely, two infinity) have an internal qualitative connection, the cardinal number of V is equivalent to the one of the interval $[0,1]$ (i.e., the cardinal number of \mathbb{R}), the cardinal number of X is the same as \mathbb{Q} , \mathbb{R} and \mathbb{Q} are the relation of the power-set property, so there is an internal qualitative connection between V and X , so their measures can be compared in size. Therefore, although both the measures of V and X are *infinitely small*, they are at different levels of infinite; the measure of V is an *infinitely small* relative to the measure 1 of the interval $[0,1]$, while the measure of X is a lower rank *infinitely small*.

Thus, under the condition that *the point is still a line segment*, V is still a measurable set. While the measures of all sets are relative, namely the measure of the interval $[0,1]$ is also relative rather than absolute.

The second explanation of Dialectical Infinity (that is, the line segment is independent of the point, time is independent of moment): Since the line segment has nothing to do with the point, there is no so-called measure in any of the point sets, and the formulation of the non-measurable set is meaningless.

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