Novel doubly periodic and other solutions of the conformable time-fractional (2+1)-dimensional modified KdV-CBS equation

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Abstract. In this paper, we aim to derive new explicit expressions for the exact solutions of the time-fractional (2+1)-dimensional modified KdV–Calogero–Bogoyavlenskii–Schiff (mKdV–CBS) equation by employing a modified approach based on Jacobi elliptic functions. This method is particularly powerful and effective, as it yields a wide variety of solutions, including doubly periodic, hyperbolic, and periodic forms. To visualize the behavior and topological structures of these solutions, we utilize ***Mathematica*** to generate illustrative surface plots.

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1 Introduction

Many phenomena in various sciences such as mathematical physics,financial mathematics, biology, fluid mechanics, elastic media, plasma and fiber optics are modeled by nonlinear differential equations by assigning analytical expressions to their exact solutions, but it is a challenge to find these expressions, so mathematicians and physicists have been interested in creating and developing methods and algorithms that allow expressing the exact solution in an explicit expression. In recent years, many of them have emerged through important research, some of which we mention as examples, but not limited to: The Lie point symmetry method [3], the consistent Riccati expansion (CRE) method [6], the extended Jacobi elliptic function expansion method [8], Tanh method [9, 10] , (G'/G)-expansion method [11, 12], the exp($-\Phi(\xi)$)-expansion method [15], Modified Kudryashov method [18], the trilinear-linear equation based on the Painlevé analysis [20, 21], Jacobi elliptic function expansion method [22], binary Bell polynomials, its bilinear formalism, Lax pair generating technique [25], bilinear Backlund transformation, Lax pair and Darboux covariant Lax pair techniques [31], nonlocal symmetry from the truncated Painlev`e expansion [32], the truncated Painlevé expansion method [33], (G'/G,1/G)-expansion method [7, 28, 34, 35].

Among the most important equations that attracted the attention of authors and focused their studies on is the KdV equation and their variants.

Recently, a combination of the mKdV equation and the modified CBS equation called as the (2+1)-dimensional modified KdV-CBS equation was established in the form

$$\begin{aligned}
&\frac{\partial}{\partial t}u(x,y,t)+\eta\left(-4u(x,y,t)^2\frac{\partial}{\partial y}u(x,y,t)-2\frac{\partial}{\partial x}u(x,y,t)\int\frac{\partial}{\partial y}u(x,y,t)^2dx+\frac{\partial^3}{\partial x^2\partial y}u(x,y,t)\right)\\
&+\rho\left(\frac{\partial^3}{\partial x^3}u(x,y,t)-6u(x,y,t)^2\frac{\partial}{\partial x}u(x,y,t)\right)=0,
\end{aligned}\tag{1.1}$$

where η,ρ are arbitrary constants.

It is evident that for $\eta=0,\rho=1$ Eq. (1.1) reduces to the modified KdV equation

$$u_t+u_{xxx}-6u^2u_x=0.$$

However, when $\eta=1,\rho=0$ the Eq.(1.1) gives the (2+1)-dimensional modified Calogero-Bogoyavlenskii-Schiff (CBS) equation

$$u_t-4u^2u_y-2u_x\int u_y^2dx+u_{xxy}=0.$$

We note that Wazwaz [33] treated the special case ($\eta=1,\rho=1$) where the truncated Painlevé expansion method was used to obtain multiple singular and complex soliton solutions. Also Y-H. Wang et al. [32] studied the same case of the equation using consistent Riccati expansion method to derive soliton and soliton-cnoidal wave solutions.

While Amin et al. [3] achieved periodic solutions by applying the Lie point symmetry method in investigating this particular case of the equation (1.1).

In this paper, we aim to study the KdV-CBS equation in its general form (η and ρ are arbitrary constants) where the derivation in time is in fractional order and in the conformable sense, so that it takes the form:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,y,t)+\eta\left(-4u(x,y,t)^2\frac{\partial}{\partial y}u(x,y,t)-2\frac{\partial}{\partial x}u(x,y,t)\int\frac{\partial}{\partial y}u(x,y,t)^2dx+\frac{\partial^3}{\partial x^2\partial y}u(x,y,t)\right)$$
$$+\rho\left(\frac{\partial^3}{\partial x^3}u(x,y,t)-6u(x,y,t)^2\frac{\partial}{\partial x}u(x,y,t)\right)=0, \quad (1.2)$$

By combining the ($G'/G;$ $1/G$)-expansion method with different Jacobi elliptic functions, new and various traveling wave solutions are obtained.

2 Preliminaries

2.1 The conformable fractional derivative

In literature, to investigate the exact solutions for fractional PDEs, the authors use many of fractional derivative sense, in particular, Riemann Liouville and Caputo derivatives

[17, 26]. In 2014, a new definition of fractional derivative is presented by R. Khalil, *et*2 *al.* [16].

Now let us give the definition and some properties of conformable fractional derivative [1, 13, 14, 15, 18, 27, 29, 30].

Definition 1. [16] Let $f:[0,+\infty)\to\mathbb{R}$ be a function. α^{th} order "conformable fractional derivative" of f is defined by

$$D_\alpha(f)(t)=\lim_{\epsilon\to0}\frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon}, \quad (2.1)$$

for all $t>0,\alpha\in(0,1)$. If f is α -differentiable in some (0,a), a>0 and $\lim_{t\to0^+}f^{(\alpha)}(t)$ then define

$f^{(\alpha)}(0)=\lim_{t\to0^+}f^{(\alpha)}(t).$

Definition 2. [14] The conformable fractional integral of order on the interval [0, t] is defined by

$$I^\alpha(f)(t)=\int_0^t s^{\alpha-1}f(s)ds. \quad (2.2)$$

This new definitions satisfies the properties which are given in the following theorems.

Theorem 2.1. [16] Let $\alpha\in(0,1]$ and f, g α - differentiable at point t>0. then

1. $D^\alpha(cf+dg)=cD^\alpha(f)+dD^\alpha(g)$ for all $c,d\in\mathbb{R}$.
2. $D^\alpha(t^p)=pt^{p-\alpha},\forall p\in\mathbb{R}.$
3. $D^\alpha(\lambda)=0$ for all constant function $f(t)=\lambda.$
4. $D^\alpha(f(t)g(t))=f(t)D^\alpha g(t)+g(t)D^\alpha f(t)$.

5. $D_t^{\alpha}(\frac{f}{g})(t) = \frac{g(t)D^{\alpha} f(t) - f(t)D^{\alpha} g(t)}{g(t)^2}$.
6. If , in addition to f is differentiable, then $D_t^{\alpha}(f)(t) = t^{1-\alpha} f'(t)$.
7. $D_t^{\alpha}(I^{\alpha} f(t)) = f(t)$.
8. $I_t^{\alpha}(D_t^{\alpha} f(t)) = f(t) - f(0)$ on the interval $[0, t]$.

Theorem 2.2. [15] Suppose $f : [0, +\infty) \to \mathbb{R}$ be a function such that f is differentiable and also α-differentiable. Let g be a function defined in the range of f and also differentiable. Then

$$D_t^{\alpha}(fog)(t) = t^{1-\alpha} g'(t) f'(g(t)).$$

Remark 1. The use of the transformation $\tau = \frac{\omega t^{\alpha}}{\alpha}$ by use of (2.) one can obtain $D_t^{\alpha} \tau = \omega$.

Furthermore, by use of (2.2) one can deduce that $D_t^{\alpha} u = u'_{\tau} D_t^{\alpha} \tau = \omega u'_{\tau}$.

Remark 2. See details for the motivation behind choosing conformable derivatives over Caputo and Riemann–Liouville types in [16].

2.2 Summary of the $(G'/G; 1/G)$ - expansion method combined with Jacobi elliptic functions

The general time-fractional nonlinear evolution equation, say in three independent variables x, y and t, is given by

$$P(u, D_t^{\alpha} u, u_x, D_{tt}^{2\alpha} u, u_{xx}, D_t^{\alpha} u_x, u_y, u_{yy}, D_t^{\alpha} u_y, u_{xy}, \ldots) = 0, \qquad (2.3)$$

Where $0 < \alpha \leq 1$, $u=u(x,t)$ is an unknown function and P is a polynomial of u and its partial fractional derivatives, in which the nonlinear terms and the highest order derivatives are included. To find the traveling wave solution of Eq. (2.3) by the $(G'/G; 1/G)$ - expansion method combined with Jacobi elliptic functions, we follow the following steps:

• **Step 1:** To obtain exact traveling wave solution, the following fractional traveling wave transformation has been applied

$$u(x,t) = U(\xi), \xi = kx + ly - \omega \frac{t^{\alpha}}{\alpha}. \qquad (2.4)$$

Where k, l and ω are constants to be discussed latter. Then, the Eq. (2.3) is reduced to the following nonlinear ordinary differential equation

$$H(U, -\omega U', kU', \omega^2 U'', k^2 U'', -k\omega U'', \lambda U, \mu U, \ldots) = 0, \qquad (2.5)$$

• **Step 2:** Assuming that the solution of Eq. (2.5) can be expressed as a finite power series of the following form

where $U^{(i)} = U_{i\xi}$

.
$$U(\xi)=a_0+\sum_{j=1}^{N}a_j\left(\frac{G'(\xi)}{G(\xi)}\right)^j+\sum_{j=1}^{N}b_j\left(\frac{G'(\xi)}{G(\xi)}\right)^{j-1}\left(\frac{1}{G(\xi)}\right), \qquad (2.6)$$

where $G=G(\xi)$ satisfies the NLODE

$$G'(\xi)^2=\lambda+\mu G(\xi)^2+\nu G(\xi)^4, \qquad (2.7)$$

which admits the following solutions ([8, 14])

1. **Case 1:** if $(\lambda=1,\mu=-(m+1),\nu=m)$, then $G(\xi)=\mathrm{sn}(\xi\,|\,m)$.

2. **Case 2:** if $(\lambda=1-m,\mu=2m-1,\nu=-m)$, then $G(\xi)=\mathrm{cn}(\xi\,|\,m)$,

3. **Case 3:** if $(\lambda=m-1,\mu=2-m,\nu=-1)$, then $G(\xi)=\mathrm{dn}(\xi\,|\,m)$,

4. **Case 4:** if $(\lambda=1,\mu=2-m,\nu=1-m)$, then $G(\xi)=\mathrm{sc}(\xi\,|\,m)$,

5. **Case 5:** if $(\lambda=1-m,\mu=2-m,\nu=1)$, then $G(\xi)=\mathrm{cs}(\xi\,|\,m)$,

6. **Case 6:** if $(\lambda=1,\mu=2m-1,\nu=m^2-m)$, then $G(\xi)=\mathrm{sd}(\xi\,|\,m)$,

7. **Case 7:** if $(\lambda=m^2-m,\mu=2m-1,\nu=1)$, then $G(\xi)=\mathrm{ds}(\xi\,|\,m)$.

Where $\mathrm{sc}(\xi\,|\,m)=\dfrac{\mathrm{sn}(\xi\,|\,m)}{\mathrm{cn}(\xi\,|\,m)}$, $\mathrm{cs}(\xi\,|\,m)=\dfrac{\mathrm{cn}(\xi\,|\,m)}{\mathrm{sn}(\xi\,|\,m)}$, $\mathrm{sd}(\xi\,|\,m)=\dfrac{\mathrm{sn}(\xi\,|\,m)}{\mathrm{dn}(\xi\,|\,m)}$, $\mathrm{ds}(\xi\,|\,m)=\dfrac{\mathrm{dn}(\xi\,|\,m)}{\mathrm{sn}(\xi\,|\,m)}$ and m is a modulus satisfies $0<m<1$.

Here, $\mathrm{sn}(\xi\,|\,m),\mathrm{cn}(\xi\,|\,m)$ and $\mathrm{dn}(\xi\,|\,m)$ are Jacobi elliptic functions [2, 19], which are double periodic and possess the following properties:

1. Some properties of the Jacobi elliptic functions:

$$\mathrm{sn}(\xi\,|\,m)^2=1-\mathrm{cn}(\xi\,|\,m)^2,$$

$$\mathrm{dn}(\xi\,|\,m)^2=1-m.\mathrm{sn}(\xi\,|\,m)^2=1-m+m.\mathrm{cn}(\xi\,|\,m)^2,$$

$$\mathrm{dc}(\xi\,|\,m)^2=(1-m)\mathrm{sc}(\xi\,|\,m)^2+1,$$

$$\mathrm{nc}(\xi\,|\,m)^2=\mathrm{sc}(\xi\,|\,m)^2+1,$$

$$\mathrm{nd}(\xi\,|\,m)^2=m\mathrm{sd}(\xi\,|\,m)^2+1,$$

$$\mathrm{cd}(\xi\,|\,m)^2=1-(1-m)\mathrm{sd}(\xi\,|\,m)^2, \qquad (2.8)$$

$$\mathrm{ns}(\xi\,|\,m)^2=\mathrm{cs}(\xi\,|\,m)^2+1,$$

$$\mathrm{ds}(\xi\,|\,m)^2=\mathrm{cs}(\xi\,|\,m)^2-m+1,$$

$$\mathrm{ns}(\xi\,|\,m)^2=\mathrm{ds}(\xi\,|\,m)^2+m,$$

$$\mathrm{cs}(\xi\,|\,m)^2=\mathrm{ds}(\xi\,|\,m)^2+m-1.$$

2. Derivatives of the Jacobi elliptic functions:

$$\frac{\partial \mathrm{sn}(\xi\,|\,m)}{\partial\xi}=\mathrm{cn}(\xi\,|\,m).\mathrm{dn}(\xi\,|\,m),$$

$$\frac{\partial \mathrm{cn}(\xi|m)}{\partial \xi} = -\mathrm{dn}(\xi|m).(\mathrm{sn})(\xi|m), \qquad (2.9)$$

$$\frac{\partial \mathrm{dn}(\xi|m)}{\partial \xi} = -m.\mathrm{sn}(\xi|m).\mathrm{cn}(\xi|m),$$

3. Some limits: we have

$$\lim_{m\to 1}(\mathrm{cn}(\xi|m)) = \mathrm{sech}(\xi), \lim_{m\to 0}(\mathrm{cn}(\xi|m)) = \cos(\xi),$$

$$\lim_{m\to 1}(\mathrm{sn}(\xi|m)) = \tanh(\xi), \lim_{m\to 0}(\mathrm{sn}(\xi|m)) = \sin(\xi),$$

$$\lim_{m\to 1}(\mathrm{dn}(\xi|m)) = \mathrm{sech}(\xi), \qquad (2.10)$$

$$\lim_{m\to 1}(\mathrm{sc}(\xi|m)) = \sinh(\xi), \lim_{m\to 0}(\mathrm{sc}(\xi|m)) = \tan(\xi),$$

$$\lim_{m\to 1}(\mathrm{cs}(\xi|m)) = \mathrm{csch}(\xi), \lim_{m\to 0}(\mathrm{cs}(\xi|m)) = \cot(\xi),$$

$$\lim_{m\to 1}(\mathrm{sd}(\xi|m)) = \sinh(\xi), \lim_{m\to 0}(\mathrm{sd}(\xi|m)) = \sin(\xi),$$

$$\lim_{m\to 1}(\mathrm{ds}(\xi|m)) = \mathrm{csch}(\xi), \lim_{m\to 0}(\mathrm{ds}(\xi|m)) = \csc(\xi).$$

• **Step 3:** The degree N of the power series (2.6) is determined by considering the homogeneous balance between the nonlinear term in Eq. (2.5) and the highest order derivative. We define the degree of $U(\xi)$ is $D[U(\xi)] = N$, which give rise to the degree of other expressions as, for example,

$$D\left[\frac{\partial^\beta U(\xi)}{\partial \xi^\beta}\right] = \beta + N, D\left[\left(\frac{\partial^\beta U(\xi)}{\partial \xi^\beta}\right)^q\right] = q(\beta + N), \qquad (2.11)$$

$$D\left[U^p(\xi)\left(\frac{\partial^\beta U(\xi)}{\partial \xi^\beta}\right)^q\right] = Np + q(\beta + N). \qquad (2.12)$$

• **Step 4:** Substituting Eqs. (2.6)-(2.7) using (2.8)-(2.9) into Eq.(2.5). Then collecting the coefficients of like powers of $\mathrm{cn}(\xi|m)^\kappa \mathrm{sn}(\xi|m)^\ell$, $\mathrm{dn}(\xi|m)^\kappa \mathrm{sn}(\xi|m)^\ell$ where $\kappa = 0,1; \ell = 0,1,2,3\ldots n$.
By setting them to zero, we get a system of algebraic equations, and Solving the over-determined system of nonlinear algebraic equations by use of ***Mathematica*** software package, we would end up with explicit expressions for $k, l, \omega, a_j, b_j, (j = 0,1,2,\ldots, N)$.

• **Step 5:** Finally, Substitute the values k, l, ω, a_j, b_j into (2.6), we obtain the exact traveling wave solutions of the fractional nonlinear evolution equation (2.5).

3 Application of the method to the Conformable Time-fractional mKdV-CBS equation

Let's consider (CTFmKdV-CBS) equation in the form

$$D_t^\alpha u + \eta\left(-4u^2 u_y - 2u_x \partial_x^{-1}(u^2)_y + u_{xxy}\right) + \rho\left(u_{xxx} - 6u^2 u_x\right) = 0, \qquad (3.1)$$

where $0<\alpha\leq 1$ and η,ρ are arbitrary constants.

We point out that $D_t^{\alpha}(.):=\frac{\partial^{\alpha}}{\partial t^{\alpha}}(.)$ and $\partial_x^{-1}(.):=\int(.)dx.$

Using the Miura transformation [4, 5, 23, 24] v_x = 2uu_y ; the equation (3.1) is transformed into the following system:

$$D_t^{\alpha}u+\eta\left(-4u^2u_y-2u_xv+u_{xxy}\right)+\rho\left(u_{xxx}-6u^2u_x\right)=0,\qquad (3.2)$$
$$v_x=2uu_y.$$

Using the fractional complex transformation (2.4), the system (3.2) is converted to the following system of nonlinear ODEs:

$$k^2(k\rho+\eta l)U^{(3)}-\left((6k\rho+4\eta l)U^2+\omega+2\eta kV\right)U'=0,\qquad (3.3)$$
$$V'=\frac{2l}{k}UU'.$$

By integrating the second equation of (3.3) with respect ξ and taking the constant of integration as zero, yield

$$V=\frac{l}{k}U^2.\qquad (3.4)$$

By substituting the (3.4) into the first equation of (3.3), we get

$$-(k\rho+\eta l)k^2U^{(3)}+(6k\rho+4\eta l+2\eta l)U^2U'+\omega U'=0.\qquad (3.5)$$

Now, integrating (3.5) with respect ξ and taking the constant of integration as zero, we have

$$\omega U+\frac{1}{3}(6k\rho+4\eta l+2\eta l)U^3-k^2(k\rho+\eta l)U''=0.\qquad (3.6)$$

Balancing U" with U^3 gives $N+2=3N$, hence $N=1$. We then suppose that (3.6) has the following formal solutions:

- **1st approach:** $U(\xi)=a_0+a_1\dfrac{\frac{\partial \mathrm{sn}(\xi\,|\,m)}{\partial\xi}}{\mathrm{sn}(\xi\,|\,m)}+\dfrac{b_1}{\mathrm{sn}(\xi\,|\,m)},$
- **2nd approach:** $U(\xi)=a_0+a_1\dfrac{\frac{\partial \mathrm{cn}(\xi\,|\,m)}{\partial\xi}}{\mathrm{cn}(\xi\,|\,m)}+\dfrac{b_1}{\mathrm{cn}(\xi\,|\,m)},$
- **3rd approach:** $U(\xi)=a_0+a_1\dfrac{\frac{\partial \mathrm{dn}(\xi\,|\,m)}{\partial\xi}}{\mathrm{dn}(\xi\,|\,m)}+\dfrac{b_1}{\mathrm{dn}(\xi\,|\,m)},$
- **4th approach:** $U(\xi)=a_0+a_1\dfrac{\frac{\partial \mathrm{sc}(\xi\,|\,m)}{\partial\xi}}{\mathrm{sc}(\xi\,|\,m)}+\dfrac{b_1}{\mathrm{sc}(\xi\,|\,m)},$
- **5th approach:** $U(\xi)=a_0+a_1\dfrac{\frac{\partial \mathrm{cs}(\xi\,|\,m)}{\partial\xi}}{\mathrm{cs}(\xi\,|\,m)}+\dfrac{b_1}{\mathrm{cs}(\xi\,|\,m)},$

• **6th approach:** $U(\xi)=a_0+a_1\dfrac{\dfrac{\partial \mathrm{sd}(\xi\,|\,m)}{\partial\xi}}{\mathrm{sd}(\xi\,|\,m)}+\dfrac{b_1}{\mathrm{sd}(\xi\,|\,m)}$,

• **7th approach:** $U(\xi)=a_0+a_1\dfrac{\dfrac{\partial \mathrm{ds}(\xi\,|\,m)}{\partial\xi}}{\mathrm{ds}(\xi\,|\,m)}+\dfrac{b_1}{\mathrm{ds}(\xi\,|\,m)}$,

1. **Using 1st approach:** Proceeding as above (Step 4) yields a set of over determined algebraic equations with respect to a_0,a_1,b_1,k,l,ω,m

$$
\begin{gathered}
-6a_1^2a_0km\rho+2a_0^3k\rho-6a_1^2a_0k\rho+2a_0^3\eta l-6a_1^2a_0\eta l-6a_1^2a_0\eta lm+a_0\omega=0,\\
6a_1b_1^2k\rho+6a_1b_1^2\eta l-2a_1k^3\rho-2a_1\eta k^2l+2a_1^3k\rho+2a_1^3\eta l=0,\\
-2a_1^3km\rho-2a_1^3k\rho+6a_0^2a_1k\rho-2a_1^3\eta l+6a_0^2a_1\eta l-2a_1^3\eta lm+a_1+\omega=0,\\
-2a_1k^3m\rho-2a_1\eta k^2lm+2a_1^3km\rho+2a_1^3\eta lm=0, \qquad (3.7)\\
6a_0b_1^2k\rho+6a_0b_1^2\eta l+6a_0a_1^2k\rho+6a_0a_1^2\eta l=0,\\
6a_0a_1^2km\rho+6a_0a_1^2\eta lm=0,\\
6a_1^2b_1k\rho+6a_1^2b_1\eta l-2b_1k^3\rho-2b_1\eta k^2l+2b_1^3k\rho+2b_1^3\eta l=0,\\
-6a_1^2b_1km\rho+6a_0^2b_1k\rho-6a_1^2b_1k\rho+6a_0^2b_1\eta l-6a_1^2b_1\eta l-6a_1^2b_1\eta lm\\
+b_1k^3m\rho+b_1k^3\rho+b_1\eta+k^2l+b_1\eta k^2lm+b_1\omega=0,\\
12a_0a_1b_1k\rho+12a_0a_1b_1\eta l=0,\\
6a_1^2b_1km\rho++6a_1^2b_1\eta lm=0.
\end{gathered}
$$

By using the ***Mathematica*** software to solving the over determined algebraic equations, we get the following results:

$$
\begin{gathered}
\left\{a_0\to 0,a_1\to 0,b_1\to \pm k,\omega\to -k^2(m+1)(k\rho+\eta l)\right\},\\
\left\{a_0\to 0,a_1\to \pm k,b_1\to 0,\omega\to 2k^2(m+1)(k\rho+\eta l)\right\},\\
\left\{a_0\to 0,a_1\to -\frac{k}{2},b_1\to \pm\frac{k}{2},\omega\to \frac{1}{2}k^2(k\rho+\eta l),m\to 0\right\}, \qquad (3.8)\\
\left\{a_0\to 0,a_1\to \frac{k}{2},b_1\to \pm\frac{k}{2},\omega\to \frac{1}{2}k^2(k\rho+\eta l),m\to 0\right\}.
\end{gathered}
$$

So we obtain the following elliptic traveling wave solutions (double periodic solutions) of EQ. (3.2)

$$
\begin{gathered}
u_{(1,1,2)}(x,y,t)=\pm k\,\frac{(\mathrm{cn}(\xi\,|\,m)\mathrm{dn}(\xi\,|\,m))}{\mathrm{sn}(\xi\,|\,m)},\\
v_{(1,1,2)}(x,y,t)=kl\left(\frac{\mathrm{cn}(\xi\,|\,m)\mathrm{dn}(\xi\,|\,m)}{\mathrm{sn}(\xi\,|\,m)}\right)^2, \qquad (3.9)
\end{gathered}
$$

Where $\xi = kx + ly - 2k^2(m+1)(k\rho + \eta l)\dfrac{t^\alpha}{\alpha}$.

And

$$u_{(1,3,4)}(x,y,t) = \frac{\pm k}{\mathrm{sn}(\xi\,|\,m)},$$
$$v_{(1,3,4)}(x,y,t) = \frac{kl}{(\mathrm{sn}(\xi\,|\,m))^2}, \tag{3.10}$$

Where $\xi = kx + ly + k^2(m+1)(k\rho + \eta l)\dfrac{t^\alpha}{\alpha}$.

In particular, when $m \to 0$, then we get the trigonometric function solutions (periodic solutions)

$$u_{(1,5,6)}(x,y,t) = \frac{1}{2}k\cot\left(kx+ly-\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right) \pm \frac{1}{2}k\csc\left(kx+ly-\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$
$$v_{(1,5,6)}(x,y,t) = \frac{1}{4}kl\left(\cot\left(kx+ly-\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right) \pm \csc\left(kx+ly-\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\right)^2. \tag{3.11}$$

And

$$u_{(1,7,8)}(x,y,t) = -\frac{1}{2}k\cot\left(kx+ly-\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right) \pm \frac{1}{2}k\csc\left(kx+ly-\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$
$$v_{(1,7,8)}(x,y,t) = \frac{1}{4}kl\left(\cot\left(kx+ly-\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right) \pm \csc\left(kx+ly-\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\right)^2. \tag{3.12}$$

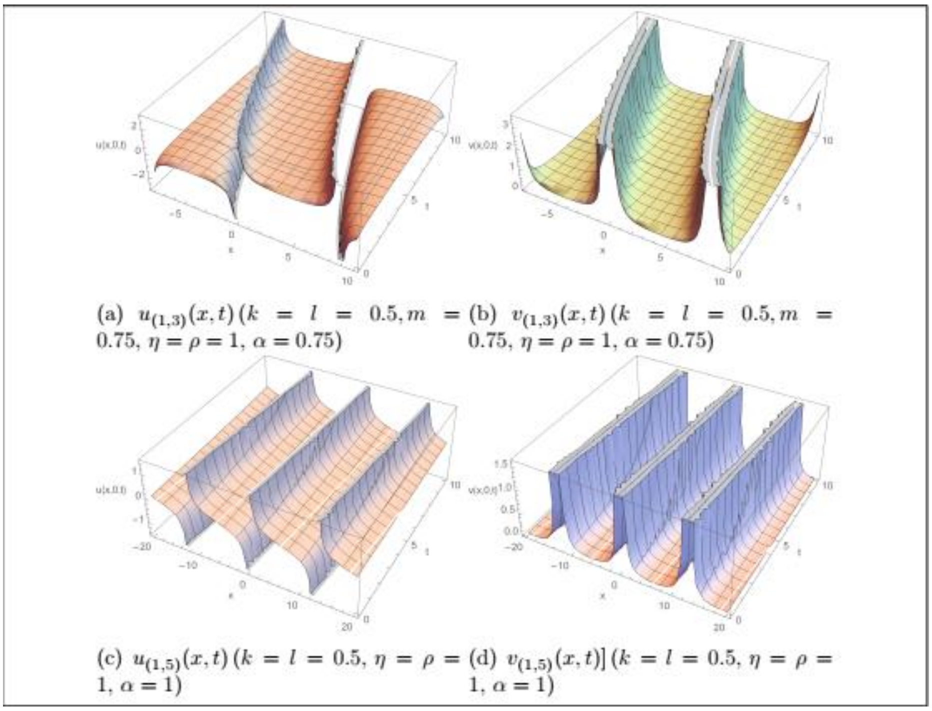


Figure 1: 3DPlot of the exact solutions of Eq. (3.2) given by (3.10) and (3.11), for $(x,t) \in [-8,10]\times[0,10]$ and $(x,t) \in [-20,20]\times[0,10]$ respectively.

In particular, when $m \to 1$, then we get the hyperbolic function solutions (solitary wave solutions) of Eq. (3.2).

$$u_{(1,9,10)}(x,y,t)=\pm k\operatorname{csch}\left(kx+ly-4k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\operatorname{sech}\left(kx+ly-4k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$

$$v_{(1,9,10)}(x,y,t)=kl\left(\operatorname{csch}\left(kx+ly-4k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\operatorname{sech}\left(kx+ly-4k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\right)^2. \quad (3.13)$$

2. **Using 2nd approach:** with the same process as above the following system of algebraic equations are obtained

$$6a_0b_1^2k\rho+6a_0b_1^2\eta l+2a_0^3k\rho+2a_0^3\eta l+a_0\omega=0,$$
$$-2a_0^3k\rho+6a_1^2a_0k\rho-2a_0^3\eta l+6a_1^2a_0\eta l-a_0\omega=0,$$
$$-6a_1b_1^2k\rho-6a_1b_1^2\eta l-4a_1k^3m\rho+2a_1k^3\rho+2a_1\eta k^2l$$
$$-4a_1\eta k^2lm-6a_0^2a_1k\rho-6a_0^2a_1\eta l-a_1\omega=0,$$
$$4a_1k^3m\rho+4a_1\eta k^2lm-2a_1^3k\rho+6a_0^2a_1k\rho-2a_1^3\eta l+6a_0^2a_1\eta l+a_1\omega=0,$$
$$-2a_1k^3m\rho-2a_1\eta k^2lm+2a_1^3km\rho+2a_1^3\eta lm=0,$$
$$-6a_0a_1^2km\rho-6a_0a_1^2\eta lm=0, \quad (3.14)$$
$$6a_0^2b_1k\rho+6a_0^2b_1\eta l+b_1k^3(-\rho)-b_1\eta k^2l+2b_1^3k\rho+2b_1^3\eta l+b_1\omega=0,$$
$$-6a_0^2b_1k\rho+6a_1^2b_1k\rho-6a_0^2b_1\eta l+6a_1^2b_1\eta l+2b_1k^3m\rho$$
$$-b_1k^3\rho-b_1\eta k^2l+2b_1\eta k^2lm-b_1\omega=0,$$
$$-12a_0a_1b_1k\rho-12a_0a_1b_1\eta l=0,$$
$$-6a_1^2b_1km\rho-6a_1^2b_1\eta lm=0.$$

By the help of ***Mathematica*** the following results are obtained

$$\left\{a_0\to 0, a_1\to 0, b_1\to\pm\sqrt{k^2-k^2m}, \omega\to -k^2(1-2m)(k\rho+\eta l)\right\},$$
$$\left\{a_0\to 0, a_1\to\pm k, b_1\to 0, \omega\to -2k^2(2m-1)(k\rho+\eta l)\right\},$$
$$\left\{a_0\to 0, a_1\to -\frac{k}{2}, b_1\to\pm\frac{k}{2}, \omega\to\frac{1}{2}k^2(k\rho+\eta l), m\to 0\right\},$$
$$\left\{a_0\to 0, a_1\to\frac{k}{2}, b_1\to\pm\frac{k}{2}, \omega\to\frac{1}{2}k^2(k\rho+\eta l), m\to 0\right\}. \quad (3.15)$$

So we obtain the following elliptic traveling wave solutions (double periodic solutions) of EQ. (3.2)

$$u_{(2,1,2)}(x,y,t)=\frac{\pm k(\operatorname{dn}(\xi\mid m)\operatorname{sn}(\xi\mid m))}{\operatorname{cn}(\xi\mid m)},$$
$$v_{(2,1,2)}(x,y,t)=\mathrm{kl}\left(\frac{\operatorname{dn}(\xi\mid m)\operatorname{sn}(\xi\mid m)}{\operatorname{cn}(\xi\mid m)}\right)^2, \quad (3.16)$$

where $\xi = kx + ly + 2k^2(2m-1)(k\rho + \eta l)\dfrac{t^\alpha}{\alpha}$.

And

$$u_{(2,3,4)}(x,y,t) = \pm \frac{k\sqrt{1-m}}{\mathrm{cn}(\xi \mid m)},$$

$$v_{(2,3,4)}(x,y,t) = \frac{\mathrm{kl}(1-m)}{\mathrm{cn}(\xi \mid m)^2}, \tag{3.17}$$

where $\xi = kx + ly + k^2(1-2m)(k\rho + \eta l)\dfrac{t^\alpha}{\alpha}$.

In particular, when $m \to 0$, then we get the trigonometric function solutions (periodic solutions) of Eq.(3.2)

$$u_{(2,5,6)}(x,y,t) = \frac{1}{2}k\tan\left(kx + ly - \frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right) \pm \frac{1}{2}k\sec\left(kx + ly - \frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$

$$v_{(2,5,6)}(x,y,t) = \frac{1}{4}\mathrm{kl}\left(\tan\left(kx + ly - \frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right) \pm \sec\left(kx + ly - \frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\right)^2, \tag{3.18}$$

and

$$u_{(2,7,8)}(x,y,t) = -\frac{1}{2}k\tan\left(kx + ly - \frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right) \pm \frac{1}{2}k\sec\left(kx + ly - \frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$

$$v_{(2,7,8)}(x,y,t) = \frac{1}{4}\mathrm{kl}\left(\tan\left(kx + ly - \frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right) \pm \sec\left(kx + ly - \frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\right)^2, \tag{3.19}$$

In particular, when $m \to 1$, then we get the hyperbolic function solutions (solitary wave solutions) of Eq.(3.2)

$$u_{(2,9,10)}(x,y,t) = \pm k\tanh\left(kx + ly + 2k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$

$$v_{(2,9,10)}(x,y,t) = kl\tanh^2\left(kx + ly + 2k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right). \tag{3.20}$$

3. **Using 3rd approach:** By inserting the 3rd equation of Eq. (2.6) along with Eq. (2.7) in Eq. (3.6) and equating the coefficient of each power of $\mathrm{cn}(\xi \mid m)^\kappa \mathrm{sn}(\xi \mid m)^\ell$, $\mathrm{dn}(\xi \mid m)^\kappa \mathrm{sn}(\xi \mid m)^\ell$ where $\kappa = 0,1; \ell = 0,1,2,3...$ to zero, we get a system of algebraic equations in different parameters a_0, a_1, b_1, k, l, m.

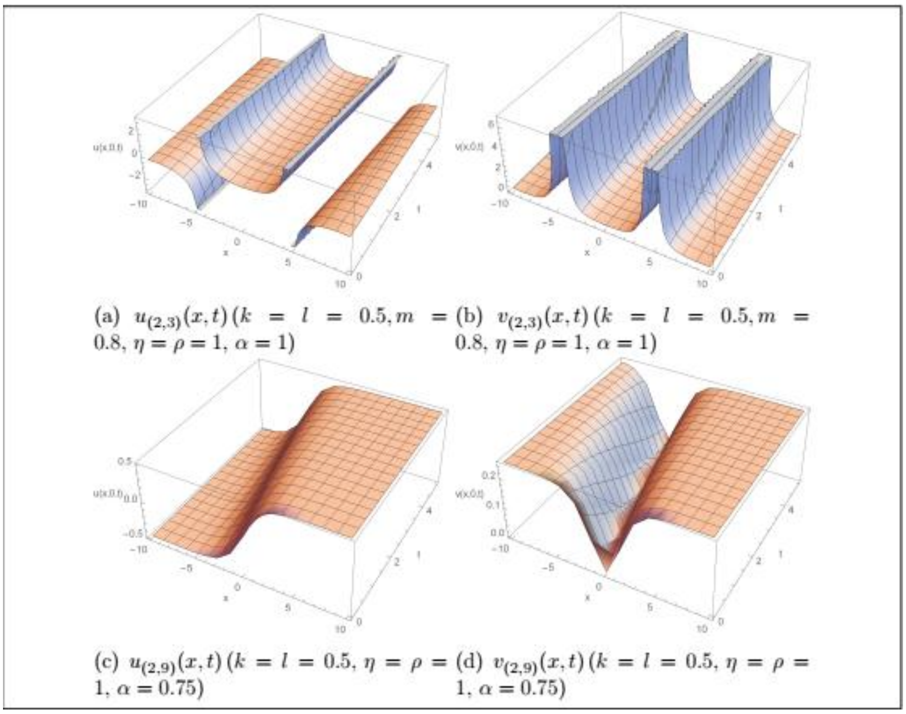


Figure 2: 3DPlot of the exact solutions of Eq. (3.2) given by (3.17) and (3.20) respectively, for $(x,t)\in[-10,10]\times[0,5]$.

$$6a_0b_1^2k\rho+6a_0b_1^2\eta l+2a_0^3k\rho+2a_0^3\eta l+a_0\omega=0,$$

$$6a_1^2a_0km^2\rho-2a_0^3km\rho+6a_1^2a_0\eta lm^2-2a_0^3\eta lm-a_0m\omega=0,$$

$$-6a_1b_1^2km\rho-6a_1b_1^2\eta lm+2a_1k^3m^2\rho-4a_1k^3m\rho+2a_1\eta k^2lm^2$$

$$-4a_1\eta k^2lm-6a_0^2a_1km\rho-6a_0^2a_1\eta lm-a_1m\omega=0,$$

$$4a_1k^3m^2\rho++4a_1\eta k^2lm^2-2a_1^3km^3\rho+6a_0^2a_1km^2\rho-2a_1^3\eta lm^3+6a_0^2a_1\eta lm^2+a_1m^2\omega=0,$$

$$-2a_1k^3m^3\rho-2a_1\eta k^2lm^3+2a_1^3km^3\rho+2a_1^3\eta lm^3=0,$$

$$-6a_0a_1^2km^2\rho-6a_0a_1^2\eta lm^2=0, \tag{3.21}$$

$$6a_0^2b_1k\rho+6a_0^2b_1\eta l+b_1k^3(-m)\rho-b_1\eta k^2lm+2b_1^3k\rho+2b_1^3\eta l+b_1\omega=0,$$

$$6a_1^2b_1km^2\rho-6a_0^2b_1km\rho+6a_1^2b_1\eta lm^2-6a_0^2b_1\eta lm+b_1k^3\left(-m^2\right)\rho$$

$$+2b_1k^3m\rho-b_1\eta k^2lm^2++2b_1\eta k^2lm-b_1m\omega=0,$$

$$-12a_0a_1b_1km\rho-12a_0a_1b_1\eta lm=0,$$

$$-6a_1^2b_1km^2\rho-6a_1^2b_1\eta lm^2=0.$$

The obtained system of algebraic equations is then solved by ***Mathematica*** to find the values of the unknown parameters $a_0, a_1, b_1, k, \omega, \lambda, \mu$.

$$\left\{a_0\to 0, a_1\to 0, b_1\to \pm ki\sqrt{1-m}, \omega\to -k^2(m-2)(k\rho+\eta l)\right\},$$

$$\left\{a_0 \to 0, a_1 \to \pm k, b_1 \to 0, \omega \to 2k^2(m-2)(k\rho+\eta l)\right\}. \tag{3.22}$$

Finally, new elliptic traveling wave solutions (double periodic solutions) for the nonlinear Eq. (3.1) are reached.

$$u_{(3,1,2)}(x,y,t) = \frac{k \pm m(\mathrm{cn}(\xi\,|\,m)\mathrm{sn}(\xi\,|\,m))}{\mathrm{dn}(\xi\,|\,m)},$$

$$v_{(3,1,2)}(x,y,t) = klm^2\left(\frac{\mathrm{cn}(\xi\,|\,m)\mathrm{sn}(\xi\,|\,m)}{\mathrm{dn}(\xi\,|\,m)}\right)^2, \tag{3.23}$$

where $\xi = kx + ly - 2k^2(m-2)(k\rho+\eta l)\dfrac{t^\alpha}{\alpha}$.

And

$$u_{(3,3,4)}(x,y,t) = \pm\frac{ik\sqrt{1-m}}{\mathrm{dn}(\xi\,|\,m)},$$

$$v_{(3,3,4)}(x,y,t) = -\frac{kl(1-m)}{\mathrm{dn}(\xi\,|\,m)^2}, \tag{3.24}$$

where $\xi = kx + ly + k^2(m-2)(k\rho+\eta l)\dfrac{t^\alpha}{\alpha}$.

4. **Using 4th approach:** following the steps indicated above we get the following system of algebraic equations

$$\begin{gathered}
-6a_1^2a_0km\rho + 2a_0^3k\rho + 12a_1^2a_0k\rho + 2a_0^3\eta l + 12a_1^2a_0\eta l - 6a_1^2a_0\eta lm + a_0\omega = 0,\\
6a_1b_1^2k\rho + 6a_1b_1^2\eta l - 2a_1k^3\rho - 2a_1\eta k^2 l + 2a_1^3k\rho + 2a_1^3\eta l = 0,\\
-2a_1^3km\rho + 4a_1^3k\rho + 6a_0^2a_1k\rho + 4a_1^3\eta l + 6a_0^2a_1\eta l - 2a_1^3\eta lm + a_1 + \omega = 0,\\
2a_1k^3m\rho - 2a_1k^3\rho - 2a_1\eta k^2 l + 2a_1\eta k^2 lm - 2a_1^3km\rho + 2a_1^3k\rho + 2a_1^3\eta l - 2a_1^3\eta lm = 0,\\
6a_0b_1^2k\rho + 6a_0b_1^2\eta l + 6a_0a_1^2\eta l = 0,\\
-6a_0a_1^2km\rho + 6a_0a_1^2k\rho + 6a_0a_1^2\eta l - 6a_0a_1^2\eta lm = 0,\\
6a_1^2b_1k\rho + 6a_1^2b_1\eta l - 2b_1k^3\rho - 2b_1\eta k^2 l + 2b_1^3k\rho + 2b_1^3\eta l = 0,\\
-6a_1^2b_1km\rho + 6a_0^2b_1k\rho + 12a_1^2b_1k\rho + 6a_0^2b_1\eta l + 12a_1^2b_1\eta l - 6a_1^2b_1\eta lm\\
+b_1k^3m\rho - 2b_1k^3\rho - 2b_1\eta k^2 l + b_1\eta k^2 lm + b_1\omega = 0,\\
12a_0a_1b_1k\rho + 12a_0a_1b_1\eta l = 0,\\
-6a_1^2b_1km\rho + 6a_1^2b_1k\rho + 6a_1^2b_1\eta l - 6a_1^2b_1\eta lm = 0.
\end{gathered} \tag{3.25}$$

Solving the system with the help of ***Mathematica*** , we get

$$\left\{a_0 \to 0, a_1 \to 0, b_1 \to \pm k, \omega \to -k^2(m-2)(k\rho+\eta l)\right\},$$

$$\left\{a_0 \to 0, a_1 \to \pm k, b_1 \to 0, \omega \to 2k^2(m-2)(k\rho+\eta l)\right\}. \tag{3.26}$$

So the following elliptic traveling wave solutions (double periodic solutions) of EQ. (3.2) are obtained

$$u_{(4,1,2)}(x,y,t) = \pm \frac{k(\mathrm{dc}(\xi \mid m)\mathrm{nc}(\xi \mid m))}{\mathrm{sc}(\xi \mid m)},$$

$$v_{(4,1,2)}(x,y,t) = kl\left(\frac{\mathrm{dc}(\xi \mid m)\mathrm{nc}(\xi \mid m)}{\mathrm{sc}(\xi \mid m)}\right)^2, \tag{3.27}$$

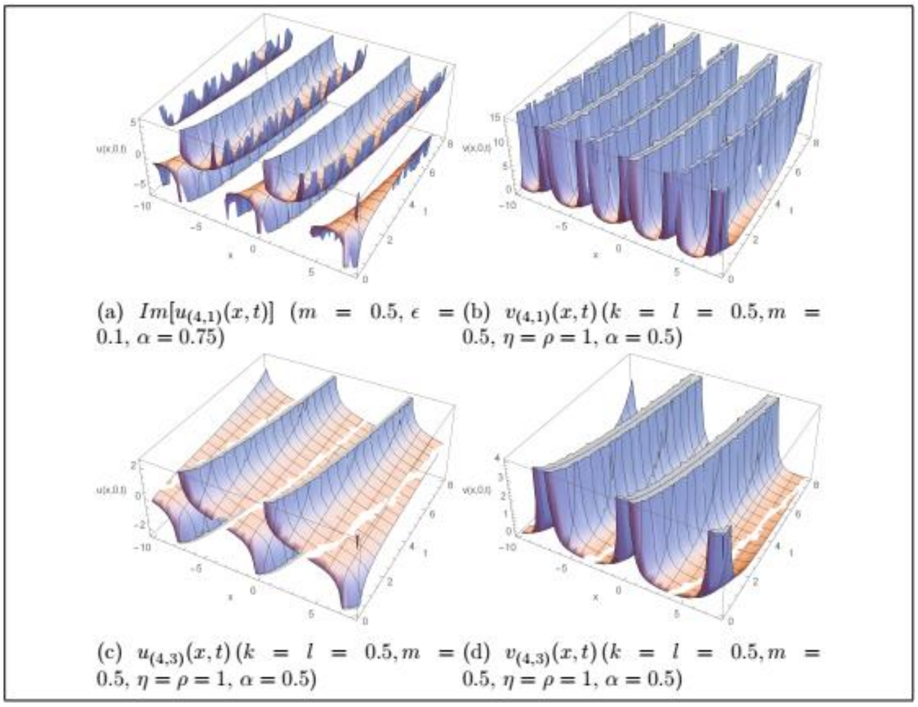
where $\xi = kx + ly - 2k^2(m-2)(k\rho+\eta l)\dfrac{t^\alpha}{\alpha}$.

And

$$u_{(4,3,4)}(x,y,t) = \pm \frac{k}{\mathrm{sc}(\xi \mid m)},$$

$$v_{(4,3,4)}(x,y,t) = \frac{kl}{\mathrm{sc}(\xi \mid m)^2}, \tag{3.28}$$

where $\xi = kx + ly + 2k^2(m-2)(k\rho+\eta l)\dfrac{t^\alpha}{\alpha}$



(a) $Im[u_{(4,1)}(x,t)]$ $(m = 0.5, \epsilon = 0.1, \alpha = 0.75)$

(b) $v_{(4,1)}(x,t)$ $(k = l = 0.5, m = 0.5, \eta = \rho = 1, \alpha = 0.5)$

(c) $u_{(4,3)}(x,t)$ $(k = l = 0.5, m = 0.5, \eta = \rho = 1, \alpha = 0.5)$

(d) $v_{(4,3)}(x,t)$ $(k = l = 0.5, m = 0.5, \eta = \rho = 1, \alpha = 0.5)$

Figure 3: 3DPlot of the exact solutions of Eq. (3.2) given by (3.27) and (3.28), for $(x,t) \in [-10,8] \times [0,8]$.

In particular, when $m \to 0$, then we get the trigonometric function solutions (periodic solutions) of Eq.(3.2)

$$u_{(4,5,6)}(x,y,t)=\pm k\cot\left(+kx+ly-2k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$

$$v_{(4,5,6)}(x,y,t)=kl\cot^2\left(kx+ly-2k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right), \qquad (3.29)$$

and, when $m\to 1$, then we get the hyperbolic function solutions (solitary wave solutions) of Eq.(3.2)

$$u_{(4,7,8)}(x,y,t)=\pm k\coth\left(kx+ly+2k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$

$$v_{(4,7,8)}(x,y,t)=kl\coth^2\left(kx+ly+2k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right), \qquad (3.30)$$

5. **Using 5th approach:** following the steps indicated above we get the following system of algebraic equations

$$\begin{gathered}
-6a_1^2a_0km\rho+2a_0^3k\rho+12a_1^2a_0k\rho+2a_0^3\eta l+12a_1^2a_0\eta l-6a_1^2a_0\eta lm+a_0\omega=0,\\
-6a_1b_1^2k\rho-6a_1b_1^2\eta l-2a_1k^3m\rho+2a_1k^3\rho+2a_1\eta k^2l\\
-2a_1\eta k^2lm+2a_1^3km\rho-2a_1^3k\rho-2a_1^3\eta l+2a_1^3\eta lm=0,\\
2a_1^3km\rho-4a_1^3k\rho-6a_0^2a_1k\rho-4a_1^3\eta l-6a_0^2a_1\eta l+2a_1^3\eta lm-a_1\omega=0,\\
2a_1k^3\rho++2a_1\eta k^2l-2a_1^3k\rho-2a_1^3\eta l=0, \qquad (3.31)\\
6a_0b_1^2k\rho+6a_0b_1^2\eta l-6a_0a_1^2km\rho+6a_0a_1^2k\rho+6a_0a_1^2\eta l-6a_0a_1^2\eta lm=0,\\
-12a_0a_1b_1k\rho-12a_0a_1b_1\eta l=0,\\
6a_0a_1^2k\rho+6a_0a_1^2\eta l=0,\\
-6a_1^2b_1km\rho+6a_1^2b_1k\rho+6a_1^2b_1\eta l-6a_1^2b_1\eta lm+2b_1k^3m\rho\\
-2b_1k^3\rho-2b_1\eta k^2l+2b_1\eta k^2lm+2b_1^3k\rho+2b_1^3\eta l=0,\\
-6a_1^2b_1km\rho+6a_0^2b_1k\rho+12a_1^2b_1k\rho+6a_0^2b_1\eta l+12a_1^2b_1\eta l-6a_1^2b_1\eta lm\\
+b_1k^3m\rho-2b_1k^3\rho-2b_1\eta k^2l+b_1\eta k^2lm+b_1\omega=0,\\
6a_1^2b_1k\rho+6a_1^2b_1\eta l=0,
\end{gathered}$$

Solving the system with the help of ***Mathematica*** , we get

$$\left\{a_0\to 0,a_1\to 0,b_1\to -\sqrt{k^2-k^2m},\omega\to -k^2(m-2)(k\rho+\eta l)\right\},$$

$$\left\{a_0 \to 0, a_1 \to 0, b_1 \to \sqrt{k^2 - k^2 m}, \omega \to -k^2(m-2)(k\rho+\eta l)\right\},$$

$$\left\{a_0 \to 0, a_1 \to -k, b_1 \to 0, \omega \to 2k^2(m-2)(k\rho+\eta l)\right\}, \tag{3.32}$$

$$\left\{a_0 \to 0, a_1 \to k, b_1 \to 0, \omega \to 2k^2(m-2)(k\rho+\eta l)\right\}.$$

So the following elliptic traveling wave solutions (double periodic solutions) of EQ. (3.2) are obtained

$$u_{(5,1,2)}(x,y,t) = \frac{\pm k(\mathrm{ds}(\xi\,|\,m)\mathrm{ns}(\xi\,|\,m))}{\mathrm{cs}(\xi\,|\,m)},$$

$$v_{(5,1,2)}(x,y,t) = kl\left(\frac{\mathrm{ds}(\xi\,|\,m)\mathrm{ns}(\xi\,|\,m)}{\mathrm{cs}(\xi\,|\,m)}\right)^2, \tag{3.33}$$

where $\xi = kx + ly - 2k^2(m-2)(k\rho+\eta l)\dfrac{t^\alpha}{\alpha}$,

and

$$u_{(5,3,4)}(x,y,t) = \pm\frac{k}{\mathrm{cs}(\xi\,|\,m)},$$

$$v_{(5,3,4)}(x,y,t) = \frac{kl(1-m)}{\mathrm{cs}(\xi\,|\,m)^2}, \tag{3.34}$$

where $\xi = kx + ly + k^2(m-2)(k\rho+\eta l)\dfrac{t^\alpha}{\alpha}$.

In particular, when $m \to 0$, then we get the trigonometric function solutions (periodic solutions) of Eq. (3.2)

$$u_{(5,5,6)}(x,y,t) = \pm k\csc\left(kx+ly-4k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\sec\left(kx+ly-4k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$

$$v_{(5,5,6)}(x,y,t) = kl\left(\csc\left(kx+ly-4k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\sec\left(kx+ly-4k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\right)^2, \tag{3.35}$$

and

$$u_{(5,7,8)}(x,y,t) = \pm k\tan\left(kx+ly-2k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$

$$v_{(5,7,8)}(x,y,t)=kl\tan^2\left(kx+ly-2k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right). \quad (3.36)$$

6. **Using 6th approach:** following the steps indicated above we get the following system of algebraic equations

$$12a_1^2a_0km\rho+2a_0^3k\rho-6a_1^2a_0k\rho+2a_0^3\eta l-6a_1^2a_0\eta l+12a_1^2a_0\eta lm+a_0\omega=0,$$

$$6a_1b_1^2k\rho+6a_1b_1^2\eta l-2a_1k^3\rho-2a_1\eta k^2l+2a_1^3k\rho+2a_1^3\eta l=0,$$

$$4a_1^3km\rho-2a_1^3k\rho+6a_0^2a_1k\rho-2a_1^3\eta l+6a_0^2a_1\eta l+4a_1^3\eta lm+a_1+\omega=0,$$

$$-2a_1k^3m^2\rho+2a_1k^3m\rho-2a_1\eta k^2lm^2+2a_1\eta k^2lm+2a_1^3km^2\rho$$

$$-2a_1^3km\rho+2a_1^3\eta lm^2-2a_1^3\eta lm=0, \quad (3.37)$$

$$6a_0b_1^2k\rho+6a_0b_1^2\eta l+6a_0a_1^2k\rho+6a_0a_1^2\eta l=0,$$

$$12a_0a_1b_1k\rho+12a_0a_1b_1\eta l=0,$$

$$6a_0a_1^2km^2\rho-6a_0a_1^2km\rho+6a_0a_1^2\eta lm^2-6a_0a_1^2\eta lm=0,$$

$$6a_1^2b_1k\rho+6a_1^2b_1\eta l-2b_1k^3\rho-2b_1\eta k^2l+2b_1^3k\rho+2b_1^3\eta l=0,$$

$$12a_1^2b_1km\rho+6a_0^2b_1k\rho-6a_1^2b_1k\rho+6a_0^2b_1\eta l-6a_1^2b_1\eta l$$

$$+12a_1^2b_1\eta lm-2b_1k^3m\rho+b_1k^3\rho+b_1\eta+k^2l-2b_1\eta k^2lm+b_1\omega=0,$$

$$6a_1^2b_1km^2\rho-6a_1^2b_1km\rho+6a_1^2b_1\eta lm^2-6a_1^2b_1\eta lm=0.$$

Solving the system with the help of ***Mathematica*** , we get

$$\left\{a_0\to 0,a_1\to 0,b_1\to \pm k,\omega\to k^2(2m-1)(k\rho+\eta l)\right\},$$

$$\left\{a_0\to 0,a_1\to \pm k,b_1\to 0,\omega\to -2k^2(2m-1)(k\rho+\eta l)\right\}. \quad (3.38)$$

So the following elliptic traveling wave solutions (double periodic solutions) of EQ. (3.2) are obtained

$$u_{(6,1,2)}(x,y,t)=\frac{\pm k(\mathrm{cd}(\xi\mid m)\mathrm{nd}(\xi\mid m))}{\mathrm{sd}(\xi\mid m)},$$

$$v_{(6,1,2)}(x,y,t)=\mathrm{kl}\left(\frac{\mathrm{cd}(\xi\mid m)\mathrm{nd}(\xi\mid m)}{\mathrm{sd}(\xi\mid m)}\right)^2, \quad (3.39)$$

where $\xi=kx+ly+2k^2(2m-1)(k\rho+\eta l)\dfrac{t^\alpha}{\alpha}$, and

$$u_{(6,3,4)}(x,y,t)=\pm\frac{k}{\mathrm{sd}(\xi\,|\,m)},$$

$$v_{(6,3,4)}(x,y,t)=\frac{\mathrm{kl}}{\mathrm{sd}(\xi\,|\,m)^{2}},\tag{3.40}$$

where $\xi = kx+ly-k^{2}(2m-1)(k\rho+\eta l)\dfrac{t^{\alpha}}{\alpha}$.

In particular, when $m\to 0$, then we get the trigonometric function solutions (periodic solutions) of Eq.(3.2)

$$u_{(6,5,6)}(x,y,t)=\pm k\csc\left(kx+ly+k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right),$$

$$v_{(6,5,6)}(x,y,t)=kl\csc^{2}\left(kx+ly+k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right).\tag{3.41}$$

and

$$u_{(6,7,8)}(x,y,t)=\frac{1}{2}k\cot\left(kx+ly-\frac{1}{2}k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right)\pm\frac{1}{2}k\csc\left(kx+ly-\frac{1}{2}k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right),$$

$$v_{(6,7,8)}(x,y,t)=\frac{1}{4}\mathrm{kl}\left(\cot\left(kx+ly-\frac{1}{2}k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right)\pm\csc\left(kx+ly-\frac{1}{2}k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right)\right)^{2},\tag{3.42}$$

and

$$u_{(6,9,10)}(x,y,t)=-\frac{k}{2}\cot\left(kx+ly-\frac{1}{2}k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right)\pm\frac{k}{2}\csc\left(kx+ly-\frac{1}{2}k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right),$$

$$v_{(6,9,10)}(x,y,t)=\frac{\mathrm{kl}}{4}\left(\cot\left(kx+ly-\frac{1}{2}k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right)\pm\csc\left(kx+ly-\frac{1}{2}k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right)\right)^{2}.\tag{3.43}$$

When $m\to 1$, then we get the hyperbolic function solutions (solitary wave solutions) of Eq.(3.2)

$$u_{(6,11,12)}(x,y,t)=\pm k\,\mathrm{csch}\left(kx+ly-k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right),$$

$$v_{(6,11,12)}(x,y,t)=kl\,\mathrm{csch}^{2}\left(kx+ly-k^{2}(k\rho+\eta l)\frac{t^{\alpha}}{\alpha}\right).\tag{3.44}$$

and

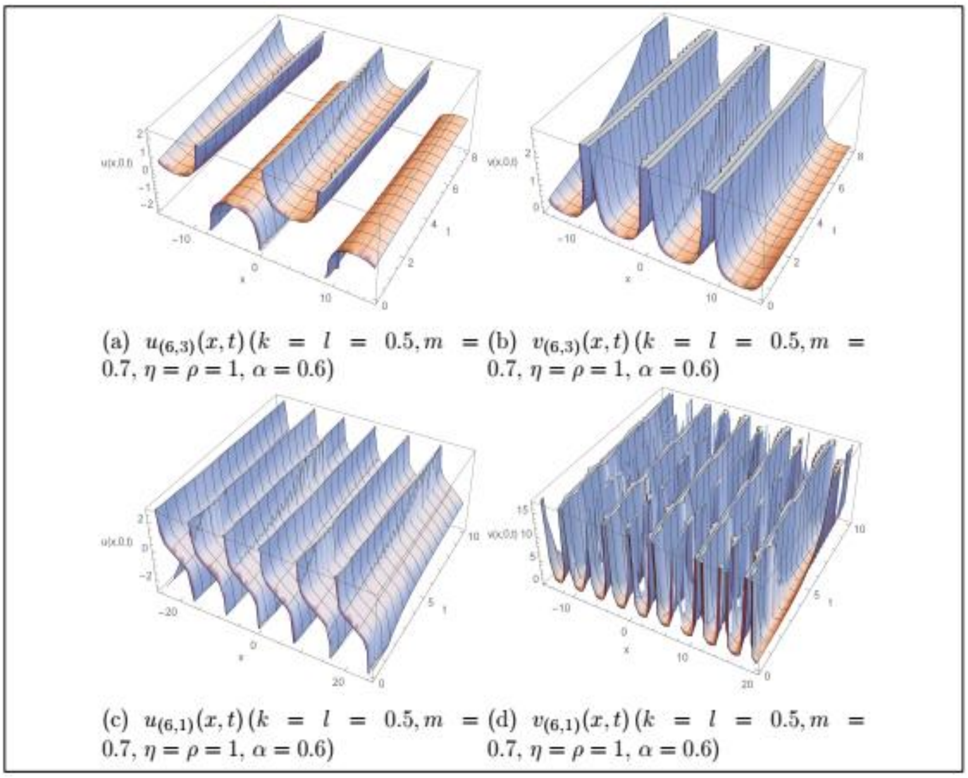
$$u_{(6,13,4)}(x,y,t)=\frac{1}{2}k\coth\left(kx+ly+\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\pm\frac{1}{2}\text{kcsch}\left(kx+ly+\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$

$$v_{(6,13,14)}(x,y,t)=\frac{1}{4}\text{kl}\left(\coth\left(kx+ly+\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\pm\text{csch}\left(kx+ly+\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\right)^2, \quad (3.45)$$

and

$$u_{(6,15,16)}(x,y,t)=-\frac{1}{2}k\coth\left(kx+ly+\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\pm\frac{1}{2}\text{kcsch}\left(kx+ly+\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right),$$

$$v_{(6,15,16)}(x,y,t)=\frac{1}{4}\text{kl}\left(\coth\left(kx+ly+\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\pm\text{csch}\left(kx+ly+\frac{1}{2}k^2(k\rho+\eta l)\frac{t^\alpha}{\alpha}\right)\right)^2. \quad (3.46)$$



(a) $u_{(6,3)}(x,t)\,(k=l=0.5, m=0.7, \eta=\rho=1, \alpha=0.6)$

(b) $v_{(6,3)}(x,t)\,(k=l=0.5, m=0.7, \eta=\rho=1, \alpha=0.6)$

(c) $u_{(6,1)}(x,t)\,(k=l=0.5, m=0.7, \eta=\rho=1, \alpha=0.6)$

(d) $v_{(6,1)}(x,t)\,(k=l=0.5, m=0.7, \eta=\rho=1, \alpha=0.6)$

Figure 4: 3DPlot of the exact solutions of Eq. (3.2) given by (3.40) and (3.39) respectively, for $(x,t)\in[-15,15]\times[0,8]$ and $(x,t)\in[-25,25]\times[0,10]$.

7. **Using 7th approach:** following the steps indicated above we get the following system of algebraic equations

$$12a_1^2a_0km\rho+2a_0^3k\rho-6a_1^2a_0k\rho+2a_0^3\eta l-6a_1^2a_0\eta l+12a_1^2a_0\eta lm+a_0\omega=0,$$

$$-6a_1b_1^2k\rho-6a_1b_1^2\eta l+2a_1k^3m^2\rho-2a_1k^3m\rho+2a_1\eta k^2lm^2-2a_1\eta k^2lm$$

$$-2a_1^3km^2\rho+2a_1^3km\rho-2a_1^3\eta lm^2+2a_1^3\eta lm=0,$$

$$-4a_1^3km\rho+2a_1^3k\rho-6a_0^2a_1k\rho+2a_1^3\eta l-6a_0^2a_1\eta l-4a_1^3\eta lm-a_1\omega=0,$$

$$2a_1k^3\rho+2a_1\eta k^2l-2a_1^3k\rho-2a_1^3\eta l=0, \tag{3.47}$$

$$6a_0b_1^2k\rho+6a_0b_1^2\eta l+6a_0a_1^2km^2\rho-6a_0a_1^2km\rho+6a_0a_1^2\eta lm^2-6a_0a_1^2\eta lm=0,$$

$$6a_0a_1^2k\rho++6a_0a_1^2\eta l=0,$$

$$6a_1^2b_1km^2\rho-6a_1^2b_1km\rho+6a_1^2b_1\eta lm^2-6a_1^2b_1\eta lm$$

$$-2b_1k^3m^2\rho+2b_1k^3m\rho-2b_1\eta k^2lm^2+2b_1\eta k^2lm+2b_1^3k\rho+2b_1^3\eta l=0,$$

$$12a_1^2b_1km\rho+6a_0^2b_1k\rho-6a_1^2b_1k\rho+6a_0^2b_1\eta l-6a_1^2b_1\eta l$$

$$+12a_1^2b_1\eta lm-2b_1k^3m\rho+b_1k^3\rho+b_1\eta+k^2l-2b_1\eta k^2lm+b_1\omega=0,$$

$$-12a_0a_1b_1k\rho-12a_0a_1b_1\eta l=0,$$

$$6a_1^2b_1k\rho+6a_1^2b_1\eta l=0,$$

Solving the system with the help of ***Mathematica*** , we get

$$\left\{a_0\to 0, a_1\to 0, b_1\to \pm ki\sqrt{m-m^2}, \omega\to k^2(2m-1)(k\rho+\eta l)\right\},$$

$$\left\{a_0\to 0, a_1\to \pm k, b_1\to 0, \omega\to -2k^2(2m-1)(k\rho+\eta l)\right\}. \tag{3.48}$$

So the following elliptic traveling wave solutions (double periodic solutions) of EQ. (3.2) are obtained

$$u_{(7,1,2)}(x,y,t)=\frac{\pm k(\mathrm{cs}(\xi\,|\,m)\mathrm{ns}(\xi\,|\,m))}{\mathrm{ds}(\xi\,|\,m)},$$

$$u_{(7,1,2)}(x,y,t)=kl\left(\frac{\mathrm{cs}(\xi\,|\,m)\mathrm{ns}(\xi\,|\,m)}{\mathrm{ds}(\xi\,|\,m)}\right)^2, \tag{3.49}$$

where $\xi=kx+ly+2k^2(2m-1)(k\rho+\eta l)\dfrac{t^\alpha}{\alpha}$,

and

$$u_{(7,3,4)}(x,y,t)\pm\frac{ik\sqrt{m-m^2}}{\mathrm{ds}(\xi\,|\,m)},$$

$$v_{(7,3,4)}(x,y,t)=-\frac{kl\left(m-m^2\right)}{\mathrm{ds}(\xi\,|\,m)^2}, \tag{3.50}$$

where $\xi=kx+ly-k^2(2m-1)(k\rho+\eta l)\dfrac{t^\alpha}{\alpha}$.

In particular, when $m \to 0$, then we get the trigonometric function solutions (periodic solutions) of Eq. (3.2)

$$u_{(7,5,6)}(x,y,t) = \pm k \cot\left(kx + ly - 2k^2(k\rho + \eta l)\frac{t^\alpha}{\alpha} \right),$$

$$v_{(7,5,6)}(x,y,t) = kl \cot^2\left(kx + ly - 2k^2(k\rho + \eta l)\frac{t^\alpha}{\alpha} \right), \tag{3.51}$$

and, when $m \to 1$, then we get the hyperbolic function solutions (solitary wave solutions) of Eq.(3.2)

$$u_{(7,7,8)}(x,y,t) = \pm k \coth\left(kx + ly + 2k^2(k\rho + \eta l)\frac{t^\alpha}{\alpha} \right),$$

$$v_{(7,7,8)}(x,y,t) = kl \coth^2\left(kx + ly + 2k^2(k\rho + \eta l)\frac{t^\alpha}{\alpha} \right). \tag{3.52}$$

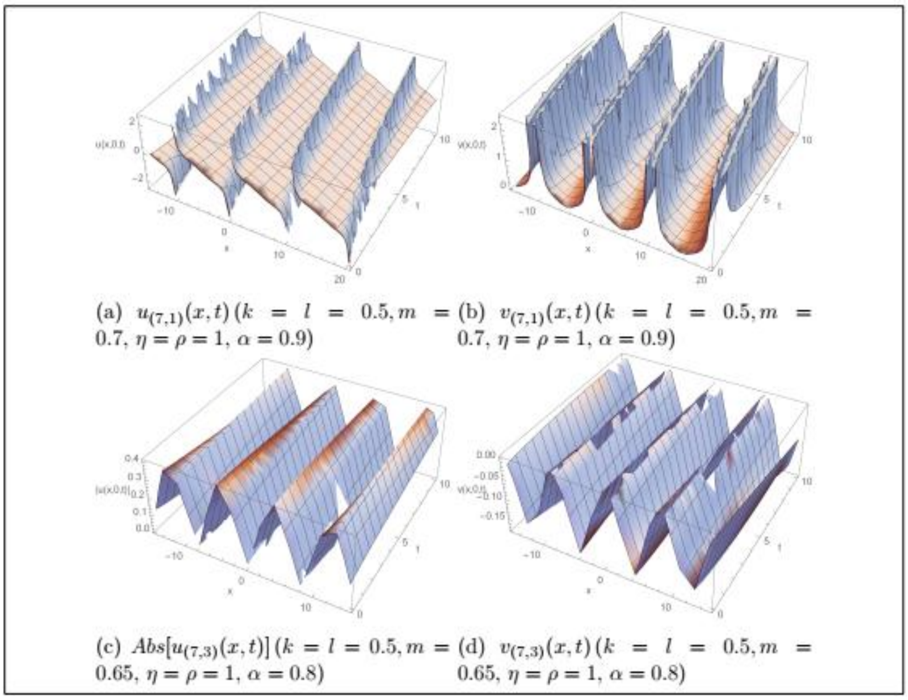


Figure 5: 3DPlot of the exact solutions of Eq. (3.2) given by (3.49) and (3.50) respectively, for $(x,t) \in [-15, 20] \times [0,10]$ and $(x,t) \in [-15,15] \times [0,10]$.

4 Conclusion

In the present article, the (*G'/G*, 1/*G*) - expansion method combined with Jacobi elliptic functions has been applied. By following the steps of the method on the conformable time-fractional (2+1)-dimensional modified KdV-Calogero-Bogoyavlenskii- Schiff equation, new exact traveling wave solutions are constructed.

Through the many and varied solutions obtained, it became clear to us that the technique used is more effective and powerful than its counterparts (Jacobi elliptic function expansion methods).

Due to its flexibility it can be applied to other partial differential equations and systems that model phenomena in various sciences.

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