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Discretization of the Alpha Power Weibull-G Family of Distributions: A Novel Discrete Distribution with Properties, Estimation, and Applications to Medical and Educational Data

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Abstract. This article introduces a new four-parameter discrete distribution, named the discrete alpha power Weibull-exponential (DAPWE) distribution. The new distribution is obtained by applying the survival discretization method to the alpha power Weibull-G family of distributions. The new distribution is highly flexible due to its ability to exhibit symmetric and asymmetric shapes of its probability mass function. Additionally, the hazard function exhibits various shapes including uniform, increasing, decreasing, J-shaped, reversed J-shaped and bathtub showing its versatility. Furthermore, some important characteristics of the proposed distribution, such as moments, order statistics and entropy are discussed. The method of maximum likelihood approach is used to estimate the distribution's unknown parameters. The efficiency of the maximum likelihood in estimating the model's parameters is assessed through simulation studies. The model performance is also evaluated through four real medical and educational data sets. The results demonstrate that the suggested distribution can indeed provide a better fit to the data compared to other distributions.

1. Introduction

Utilizing statistical distributions is vital for analyzing different types of data. Traditional distributions often fail to capture the underlying patterns and characteristics present in real-world data. This limitation has driven the creation of new distribution families aimed at better handling complex data structures. Similarly, discrete distributions play a key role in modeling product lifetimes on non-continuous scales, prompting the development of innovative discrete models. However, these also face limitations when applied to intricate data scenarios, highlighting the demand for more adaptable alternatives. Consequently, there has been a growing interest in converting existing continuous models into discrete ones to improve their fit for a wider range of data sets.

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Numerous approaches for introducing new families of statistical distributions have been documented in the literature, such as the Marshall-Olkin method [1] and the exponentiated method [2]. More recently, a new technique known as the transformed-transformer (T-X) method has been introduced by [3] to create novel distribution families. The cumulative distribution function (CDF) and the probability density function (PDF) of this family can be respectively defined by

$$F(x) = \int_{a}^{W(G(x))} r(t)dt, \tag{1.1}$$

$$f(x) = \{\frac{d}{dx}W(G(x))\}r\{W(G(x))\}. \tag{1.2}$$

where $W(G(x)) = -log(1 - G(x))$ is a weight function and $G(x)$ is the cumulative distribution function of a baseline distribution. This method has been widely used in literature to construct new families of distributions such as gamma-G, beta-exponential-G, and Weibull-G distributions.

[4] proposed a new approach that involves adding extra parameter to existing distributions, which enhances the richness and adaptability of the resulting distribution for data modeling. This approach is known as the alpha power transformation (APT). The APT method was utilized by various researchers, such as [5], [6], [7] and [8].

The APT family has the following CDF and PDF ,respectively.

$$F(x) = \begin{cases} \frac{\alpha^{F(x)} - 1}{\alpha - 1} & \text{if } \alpha > 0, \alpha \neq 1 \\ F(x) & \text{if } \alpha = 1 \end{cases} \tag{1.3}$$

$$f(x) = \begin{cases} \frac{\log \alpha}{\alpha - 1} f(x)\alpha^{F(x)} & \text{if } \alpha > 0,\ \alpha \neq 1 \\ f(x) & \text{if } \alpha = 1 \end{cases} \tag{1.4}$$

[9] presented a new approach to generate distributions by combining two techniques, the T-X method proposed by [3] and the alpha power transformation (APT) approach proposed by [4], which significantly increases the flexibility of the resulting distributions. Their proposed family is called the alpha power Weibull-G family of distributions and has the following CDF

$$F(x) = \begin{cases} \dfrac{\alpha^{1-\exp\left(-\left(\frac{-\log(1-G(x,\delta))}{\beta}\right)^{c}\right)} - 1}{\alpha - 1}, & \alpha > 0,\ \alpha \neq 1, \\ 1 - \exp\left(-\left(\dfrac{-\log(1-G(x,\delta))}{\beta}\right)^{c}\right), & \alpha = 1. \end{cases} \tag{1.5}$$

Thus, the PDF and the survival function (SF) take the following forms, respectively.

$$f(x)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}\,\dfrac{c}{\beta}\,\dfrac{g(x)}{1-G(x)}\left(\dfrac{-\log\big(1-G(x,\delta)\big)}{\beta}\right)^{c-1} & \alpha>0,\ \alpha\neq 1,\\ \quad\times\exp\left[-\left(\dfrac{-\log\big(1-G(x,\delta)\big)}{\beta}\right)^{c}\right]\alpha^{1-\exp\left[-\left(\frac{-\log(1-G(x,\delta))}{\beta}\right)^{c}\right]} & \\ \dfrac{c}{\beta}\,\dfrac{g(x)}{1-G(x)}\left(\dfrac{-\log\big(1-G(x,\delta)\big)}{\beta}\right)^{c-1}\exp\left[-\left(\dfrac{-\log\big(1-G(x,\delta)\big)}{\beta}\right)^{c}\right] & \alpha=1.\end{cases} \tag{1.6}$$

$$S(x)=\begin{cases}\dfrac{\alpha}{\alpha-1}\left(1-\alpha^{-\exp\left(-\left(\frac{-\log(1-G(x,\delta))}{\beta}\right)^{c}\right)}\right), & \alpha>0,\ \alpha\neq 1,\\ \exp\left(-\left(\dfrac{-\log\big(1-G(x,\delta)\big)}{\beta}\right)^{c}\right), & \alpha=1.\end{cases} \tag{1.7}$$

In recent decades, various discretized versions of continuous distributions have been developed to represent diverse discrete data. The survival discretization technique introduced by [10], is one of the most significant methods of discretization. This innovative approach creates a new discrete distribution by utilizing the survival function of any continuous distribution. According to [10], The survival function (SF) of the continuous distribution can be used to define the PMF of a discrete distribution as follows:

$$P(X=x)=S(x)-S(x+1), \qquad x=0,1,2,..., \tag{1.8}$$

and

$$S(x)=P(X\geq x)=1-F(x;\eta), \tag{1.9}$$

with $F(x;\eta)$ being the CDF of the continuous distribution and η denoting a parameter vector.

Recently, new discrete families of distributions have been created and derived from existing continuous families by employing this statistical method. For instance, [11] studied a discrete analogue of the Weibull-G family, where [12] introduced the discrete odd Perks exponential distribution derived from the odd Perk-G family. [13] studied the discrete odd Weibull inverse-Weibull and discrete odd Weibull-Geometric that were proposed from a discrete version of odd Weibull-G family. Finally, [14] introduced the discrete Weibull-G family and presented the discrete Weibull exponential distribution.

Additionally, [15] presented the discrete alpha power exponential distribution. [16] presented the discrete alpha power transformed exponential distribution. [17] proposed the discrete alpha power inverse Lomax distribution. [18] presented and studied the discrete alpha power inverse Weibull distribution. [19] presented the discrete alpha power inverted Kumaraswamy distribution.

The primary interest of this article is to apply the discretization method proposed by [10] to create a new discrete distribution which offers greater flexibility in data fitting. To demonstrate this, the study employs the survival discretization technique to introduce a discrete version of the alpha power Weibull-G (APW-G) family of distributions. This discrete family is called the discrete alpha power Weibull-G (DAPW-G) family, and a four-parameter distribution known as the discrete alpha power Weibull exponential (DAPWE) distribution is established with this approach.

Applying the discretization technique by [10] and the continuous alpha power Weibull-G family introduced by [9], the discrete alpha power Weibull-G (DAPW-G) family has the following CDF and SF, respectively.

$$F(x;\alpha,\beta,c,\delta)=\frac{\alpha^{1-\exp\left[-\left(\frac{-\log\left(1-G(x+1,\delta)\right)}{\beta}\right)^{c}\right]}-1}{\alpha-1},\qquad \alpha>0,\ \alpha\neq 1. \tag{1.10}$$

$$S(x)=\frac{\alpha}{\alpha-1}\left(1-\alpha^{-\exp\left[-\left(\frac{-\log\left(1-G(x,\delta)\right)}{\beta}\right)^{c}\right]}\right),\qquad \alpha>0,\ \alpha\neq 1. \tag{1.11}$$

Therefore, the probability mass function (PMF)can be expressed as

$$f(x;\alpha,\beta,c,\delta)=\frac{\alpha}{\alpha-1}\left[\alpha^{-\exp\left(-\left(\frac{-\log(1-G(x+1,\delta))}{\beta}\right)^{c}\right)}-\alpha^{-\exp\left(-\left(\frac{-\log(1-G(x,\delta))}{\beta}\right)^{c}\right)}\right],\qquad \alpha>0,\ \alpha\neq 1. \tag{1.12}$$

Based on Equations (1.11) and (1.12), the hazard rate function (HRF) takes the form

$$h(x;\alpha,\beta,c,\delta)=\frac{\alpha^{-\exp\left(-\left(\frac{-\log\left(1-G(x+1,\delta)\right)}{\beta}\right)^{c}\right)}-\alpha^{-\exp\left(-\left(\frac{-\log\left(1-G(x,\delta)\right)}{\beta}\right)^{c}\right)}}{1-\alpha^{-\exp\left(-\left(\frac{-\log\left(1-G(x,\delta)\right)}{\beta}\right)^{c}\right)}},\qquad \alpha>0,\ \alpha\neq 1. \tag{1.13}$$

The rest of the article is organized as follows: In Section 2, the DAPWE distribution is presented with various forms of its PMF and hazard function. Section 3 explores several key properties of the DAPWE distribution. Section 4 addresses the parameter estimation through the maximum likelihood (ML) method. Section 5 includes some simulation studies to assess and evaluate the efficiency of the ML in estimating the distribution parameters. Section 6 examines four applications to illustrate the effectiveness of the proposed discrete distribution. Lastly, Section 7 concludes.

2. Discrete Alpha Power Weibull Exponential Distribution (DAPWE)

The CDF of an exponential random variable X with parameter $\theta > 0$, is represented as:

$$G(x) = 1 - e^{-\theta x}, \theta > 0, x > 0. \tag{2.1}$$

The DAPWE distribution can be expressed in terms of its CDF and PMF, respectively, by substituting $G(x)$ into equations (1.10) and (1.12).

$$F(x;\alpha,\beta,c,\theta) = \frac{\alpha^{1-\exp\left[-\left(\frac{\theta(x+1)}{\beta}\right)^{c}\right]} - 1}{\alpha - 1}. \tag{2.2}$$

$$f(x;\alpha,\beta,c,\theta) = \frac{\alpha}{\alpha-1}\left[\alpha^{-\exp\left(-\left(\frac{\theta(x+1)}{\beta}\right)^{c}\right)} - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^{c}\right)}\right]. \tag{2.3}$$

Moreover, the SF and HRF are respectively obtained by:

$$S(x) = \frac{\alpha}{\alpha-1}\left[1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^{c}\right)}\right], \qquad \alpha > 0,\ \alpha \neq 1. \tag{2.4}$$

$$h(x) = \frac{\alpha^{-\exp\left(-\left(\frac{\theta(x+1)}{\beta}\right)^{c}\right)} - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^{c}\right)}}{1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^{c}\right)}}. \tag{2.5}$$

Figures 1 and 2 illustrate various plots of PMF and HRF for the DAPWE model based on selected values of parameters. As shown in Figure 1, the plots of the PMF of the DAPWE distribution exhibit different forms, including increasing, right skewed, J-shaped, symmetric, left skewed, and reversed J-shaped forms. Moreover, Figure 2 demonstrates that the HRF of the DAPWE distribution can also take on different shapes including increasing, uniform, decreasing, bathtub, and J-shaped. These findings indicate the significant versatility of the DAPWE distribution in modeling real data.

3. Mathematical Characteristics

Some characteristics of the DAPWE distribution are provided. These characteristics include the quantile, moments, entropy and order statistics.

3.1. **Quantile function.** The DAPWE distribution has the following quantile function:

$$x_q = \left\lceil \frac{\beta}{\theta}\left\{-\log\left[1 - \left(\frac{\log[(\alpha-1)q+1]}{\log\alpha}\right)\right]\right\}^{\frac{1}{c}} - 1\right\rceil \tag{3.1}$$

where the ceiling function denoted by $\lceil . \rceil$ returns the smallest integer equal to or larger than its argument.

Specifically, by setting q = 0.5, we can get the median M for DAPWE distribution. Thus, it can be as follows:

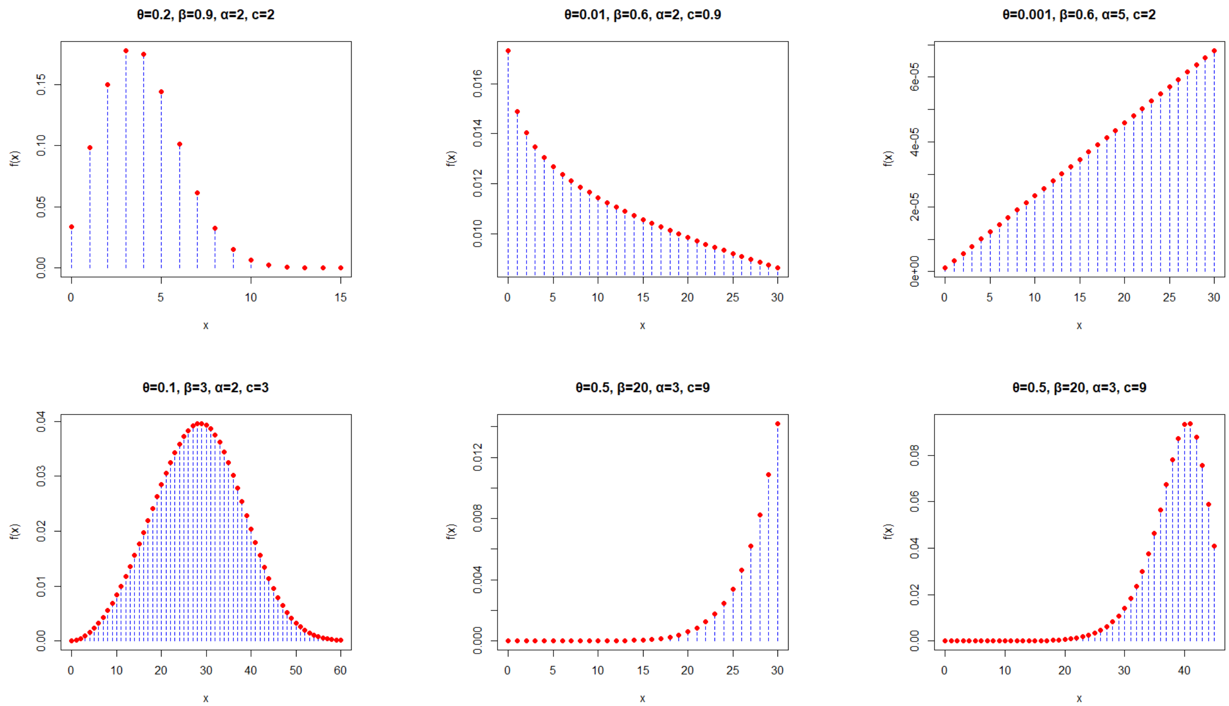

Figure 1. PMF plots of the DAPWE model.

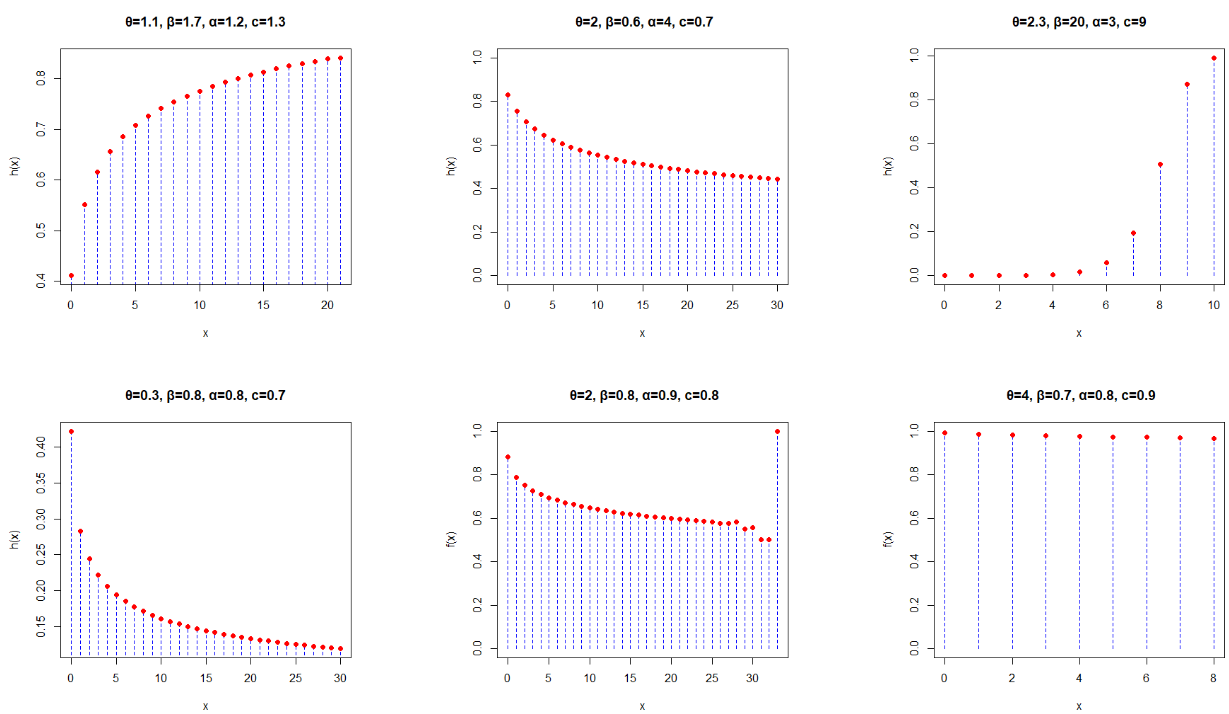

Figure 2. Hazard plots of the DAPWE model.

$$M = \left[\frac{\beta}{\theta} \left\{ -\log\left[1 - \left(\frac{\log[0.5(\alpha-1)+1]}{\log\alpha} \right) \right] \right\}^{\frac{1}{c}} - 1 \right] \tag{3.2}$$

3.2. **Moments.** For the random variable $X \sim$ DAPWE, the r^{th} moment is given by:

$$\mu_r' = E(X^r) = \sum_{x=0}^{\infty} x^r f(x) = \sum_{x=1}^{\infty} \left(x^r - (x-1)^r \right) \frac{\alpha}{\alpha-1} \left[1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)} \right]. \tag{3.3}$$

Therefore,

$$\mu_1' = E(X) = \sum_{x=1}^{\infty} \frac{\alpha}{\alpha-1} \left[1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)} \right], \qquad \alpha > 0,\ \alpha \neq 1,$$

$$\grave{\mu}_2 = E(x^2) = \sum_{x=1}^{\infty} (2x-1) \frac{\alpha}{\alpha-1} \left[1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)} \right],$$

$$\grave{\mu}_3 = E(x^3) = \sum_{x=1}^{\infty} (3x^2 - 3x + 1) \frac{\alpha}{\alpha-1} \left[1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)} \right],$$

$$\grave{\mu}_4 = E(x^4) = \sum_{x=1}^{\infty} (4x^3 - 6x^2 + 4x - 1) \frac{\alpha}{\alpha-1} \left[1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)} \right].$$

Therefore, the variance (var) of the DAPWE distribution can be expressed as:

$$\begin{aligned} \sigma^2 &= \grave{\mu}_2 - \left(\grave{\mu}_1 \right)^2 \\ &= \sum_{x=1}^{\infty} (2x-1) \frac{\alpha}{\alpha-1} \left[1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)} \right] - \left[\sum_{x=1}^{\infty} \frac{\alpha}{\alpha-1} \left(1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)} \right) \right]^2. \end{aligned}$$

Furthermore, skewness (sk) and kurtosis (K) take the following forms, respectively.

$$\text{skewness} = \frac{\grave{\mu}_3 - 3\,\grave{\mu}_2\,\grave{\mu}_1 + 2\,\grave{\mu}_1^2}{(\sigma^2)^{3/2}}$$

$$\text{kurtosis} = \frac{\grave{\mu}_4 - 4\,\grave{\mu}_3\,\grave{\mu}_1 + 6\,\grave{\mu}_2\,\grave{\mu}_1^2 - 3\,\grave{\mu}_1^4}{(\sigma^2)^2}$$

3.3. **Index of dispersion and variation coefficient.** The dispersion index (DSI) and the coefficient of variation (COV) are defined as

$$\text{DSI} = \frac{\text{var(x)}}{\text{mean}}$$

$$\text{COV(x)} = \frac{\sqrt{\text{var(x)}}}{\text{mean}}$$

Thus, the DSI and COV of the DAPWE distribution can be presented respectively as

$$\text{DSI} = \frac{\sum_{x=1}^{\infty} (2x-1) \frac{\alpha}{\alpha-1} \left[1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)} \right] - \left[\sum_{x=1}^{\infty} \frac{\alpha}{\alpha-1} \left(1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)} \right) \right]^2}{\sum_{x=1}^{\infty} \frac{\alpha}{\alpha-1} \left(1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)} \right)}.$$

$$\mathrm{COV} = \frac{\sqrt{\sum_{x=1}^{\infty}(2x-1)\,\frac{\alpha}{\alpha-1}\left[1-\alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^{c}\right)}\right]-\left[\sum_{x=1}^{\infty}\frac{\alpha}{\alpha-1}\left(1-\alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^{c}\right)}\right)\right]^{2}}}{\sum_{x=1}^{\infty}\frac{\alpha}{\alpha-1}\left(1-\alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^{c}\right)}\right)}.$$

Table 1 presents some empirical results regarding the mean, var, sk, K, DSI and COV, along with insightful commentary.

Table 1. Numerical findings for the DAPWE distribution for mean, var, sk, K, DSI and COV.

θ	β	α	c	mean	var	sk	K	DSI	COV
0.9	1	1.2	1	0.73679	1.2346690	1.905760	7.425394	1.8798239	1.552427
1.0	1	1.2	1	0.62141	0.9671538	1.945520	7.584825	1.4654336	1.379551
1.1	1	1.2	1	0.53564	0.8003945	2.008823	7.805834	1.5115865	1.490946
1.2	1	1.2	1	0.46282	0.6610743	2.077098	8.066518	1.6675849	1.946783
1.3	1	1.2	1	0.39854	0.5437588	2.127448	8.244007	1.0412824	2.127749
1.4	1	1.2	1	0.35240	0.4627927	2.174658	8.337063	1.1111111	1.924501
1.5	1	1.2	1	0.30723	0.3886625	2.244833	8.618008	1.1114852	2.028944
1.6	1	1.2	1	0.27205	0.3355876	2.321280	8.916761	0.9576409	1.757602
1.7	1	1.2	1	0.24094	0.2901822	2.398436	9.139604	1.5682916	2.611256
1	0.9	1.2	1	0.52980	0.7808929	2.000638	7.742846	1.519743	1.662278
1	1.0	1.2	1	0.62141	0.9671538	1.945520	7.584825	1.465434	1.379551
1	1.1	1.2	1	0.72257	1.2093015	1.926166	7.544507	1.844711	1.423782
1	1.2	1.2	1	0.82084	1.4472616	1.897672	7.441891	1.802483	1.473657
1	1.3	1.2	1	0.91056	1.6806824	1.871200	7.390762	1.380471	1.516834
1	1.4	1.2	1	1.02161	1.9744987	1.830342	7.170893	1.762065	1.399232
1	1.5	1.2	1	1.11304	2.2544014	1.826406	7.200551	1.556277	1.178785
1	1.6	1.2	1	1.21915	2.5761294	1.810463	7.152816	1.656490	1.116011
1	1.7	1.2	1	1.32171	2.9317738	1.794201	7.030958	2.463092	1.376476
1	1	1.2	1	0.62753	0.9865484	1.949332	7.558793	1.528139	1.557439
1	1	1.3	1	0.63933	0.9878910	1.907995	7.406633	1.465423	1.370674
1	1	1.4	1	0.65894	1.0289861	1.896712	7.357293	1.590909	1.410190
1	1	1.5	1	0.67335	1.0460567	1.880248	7.284418	1.484263	1.466665
1	1	1.6	1	0.68056	1.0418127	1.840617	7.132413	1.187238	1.701677
1	1	1.7	1	0.70166	1.0673317	1.804441	6.939396	1.441363	1.563006
1	1	1.8	1	0.70915	1.0712419	1.786681	6.851458	1.162202	1.297825
1	1	1.9	1	0.72253	1.0862054	1.761797	6.751108	1.286153	1.275947
1	1	2.0	1	0.73693	1.1023845	1.736290	6.609931	1.794797	1.568001
1	1	1.2	0.9	0.69627	1.3270573	2.168235	8.812224	1.5627706	1.670527
1	1	1.2	1.0	0.62450	0.9751560	1.932793	7.467094	2.1993171	1.760008
1	1	1.2	1.1	0.57397	0.7620835	1.731769	6.395976	1.6835017	1.675063
1	1	1.2	1.2	0.52887	0.6124845	1.566820	5.583118	0.9855700	1.326630
1	1	1.2	1.3	0.50465	0.5265062	1.419084	4.902467	1.1452991	1.484082
1	1	1.2	1.4	0.47703	0.4530415	1.282848	4.290075	0.8037518	1.280746
1	1	1.2	1.5	0.46151	0.4064951	1.174378	3.889785	0.7796836	1.212890
1	1	1.2	1.6	0.44376	0.3633758	1.058947	3.428760	0.6930085	1.165693
1	1	1.2	1.7	0.43137	0.3365266	0.982653	3.141213	0.7952250	1.344370

From the data in Table 1, it is evident that as θ increases while keeping α , β, and c constant, there is an evident decrease in both mean and variance and increase in skewness and kurtosis. Conversely, when β increases with θ , α and c remain constant, the mean and variance both rise as skewness and kurtosis decline. Similarly, increasing α while maintaining θ, β, and c leads to an increase in the mean and variance, with a decrease in skewness and kurtosis. Finally, as c increases while keeping θ, β and α constant, the mean, variance, skewness, and kurtosis all decrease.

3.4. **The moment generating function.** The moment generating function for a random variable X, where X is non-negative and follows the DAPWE distribution, can be derived as:

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} f(x) = 1 + \sum_{x=1}^{\infty} \left(e^{tx} - e^{t(x-1)}\right) \frac{\alpha}{\alpha-1} \left[1 - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)}\right].$$

3.5. **The Rényi entropy.** [20] proposed a fundamental entropy, called Rényi entropy. The entropy can be obtained by

$$\gamma_R(\delta) = \frac{1}{1-\delta} \log\left(\sum_{x=0}^{\infty} f(x)^{\delta}\right), \tag{3.4}$$

where $\delta > 0$ and $\delta \neq 1$. By substituting the equation (2.3) in entropy, we get

$$\gamma_R(\delta) = \frac{1}{1-\delta} \log\left(\sum_{x=0}^{\infty} \left[\frac{\alpha}{\alpha-1} \left(\alpha^{-\exp\left(-\left(\frac{\theta(x+1)}{\beta}\right)^c\right)} - \alpha^{-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)}\right)\right]^{\delta}\right)$$

3.6. **The order statistic.** Assume that $x_1, x_2, .., x_n$ is a random sample of DAPWE, then the i^{th} order statistic $X_{i:n}$ has the following CDF

$$F_{i:n}(x) = \sum_{r=i}^{n} \binom{n}{r} [F(x)]^r [1 - F(x)]^{n-r}, \tag{3.5}$$

using the binomial expansion equation

$$(1-y)^{k-r} = \sum_{j=0}^{k-r} (-1)^j \binom{k-r}{j} y^j,$$

therefore,

$$\begin{aligned} F_{i:n}(x) &= \sum_{r=i}^{n} \sum_{j=0}^{n-r} (-1)^j \binom{n}{r} \binom{n-r}{j} [F(x)]^{r+j} \\ &= \sum_{r=i}^{n} \sum_{j=0}^{n-r} (-1)^j \binom{n}{r} \binom{n-r}{j} \left(\frac{\alpha^{1-\exp\left(-\left(\frac{\theta(x+1)}{\beta}\right)^c\right)} - 1}{\alpha - 1}\right)^{r+j}. \end{aligned} \tag{3.6}$$

Applying the formula discussed by [21], the corresponding PMF of i^{th} order statistic is as follows:

$$f_i(y) = F_i(y) - F_i(y-1)$$

$$f_i(x) = \sum_{r=i}^{n} \sum_{j=0}^{n-r} (-1)^j \binom{n}{r} \binom{n-r}{j} \left[\left(\frac{\alpha^{1-\exp\left(-\left(\frac{\theta(x+1)}{\beta}\right)^c\right)} - 1}{\alpha - 1} \right)^{r+j} - \left(\frac{\alpha^{1-\exp\left(-\left(\frac{\theta x}{\beta}\right)^c\right)} - 1}{\alpha - 1} \right)^{r+j} \right]. \tag{3.7}$$

Particularly, the PMF of minimum and maximum order statistic are obtained by setting $i = 1$ and $i = n$ in Equation (3.7), respectively.

4. Estimation Method

The ML approach is used to estimate the distribution's parameters. For a parameter vector $\hat{\omega} = (\theta, \beta, \alpha, c)$, the ML method involves three main steps. First, the log-likelihood function must be defined. Next, the partial derivatives with respect to each parameter are calculated. Finally, these derivatives are set to zero and solved analytically.

$$L(X; \theta, \beta, \alpha, c) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{\alpha}{\alpha - 1} \left[\alpha^{-\exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right)} - \alpha^{-\exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)} \right],$$

Then, the DAPWE distribution has the following log-likelihood function

$$\ell(\theta, \beta, \alpha, c; \mathbf{x}) = \sum_{i=1}^{n} \log\left(\frac{\alpha}{\alpha - 1}\right) + \sum_{i=1}^{n} \log\left[\left(\alpha^{-\exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right)} - \alpha^{-\exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)}\right)\right] \tag{4.1}$$

The derivation of equation (4.1) with respect to the parameters gives the following results.

$$\frac{\partial \ell}{\partial \theta} = c\,\theta^{c-1} \log\alpha \sum_{i=1}^{n} \frac{\left(\frac{x_i+1}{\beta}\right)^c \exp\left[-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right] \alpha^{-\exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right)} - \left(\frac{x_i}{\beta}\right)^c \exp\left[-\left(\frac{\theta x_i}{\beta}\right)^c\right] \alpha^{-\exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)}}{\alpha^{-\exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right)} - \alpha^{-\exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)}}. \tag{4.2}$$

$$\frac{\partial \ell}{\partial \beta} = -\frac{c\,\theta^c \log\alpha}{\beta} \sum_{i=1}^{n} \frac{\left(\frac{(x_i+1)}{\beta}\right)^c \exp^{-\left(\frac{\theta(x_i+1)}{\beta}\right)^c} \alpha^{-\exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right)} - \left(\frac{x_i}{\beta}\right)^c \exp^{-\left(\frac{\theta x_i}{\beta}\right)^c} \alpha^{-\exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)}}{\alpha^{-\exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right)} - \alpha^{-\exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)}}. \tag{4.3}$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{-n}{\alpha(\alpha - 1)} - \sum_{i=1}^{n} \frac{\alpha^{-\exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right)-1} \exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right) - \alpha^{-\exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)-1} \exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)}{\alpha^{-\exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right)} - \alpha^{-\exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)}}. \tag{4.4}$$

$$\frac{\partial \ell}{\partial c} = \frac{\theta^c \log(\alpha)}{\beta^c} \sum_{i=1}^{n} \left[\frac{(x+1)^c \exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right) \alpha^{-\exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right)} \log\left(\frac{\theta(x_i+1)}{\beta}\right)}{\alpha^{-\exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right)} - \alpha^{-\exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)}} - \frac{x^c \exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right) \alpha^{-\exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)} \log\left(\frac{\theta x_i}{\beta}\right)}{\alpha^{-\exp\left(-\left(\frac{\theta(x_i+1)}{\beta}\right)^c\right)} - \alpha^{-\exp\left(-\left(\frac{\theta x_i}{\beta}\right)^c\right)}} \right]. \tag{4.5}$$

Consequently, by setting equations (4.2 - 4.5) to zero and find the solution analytically or employing numerical methods like the Newten Raphson, the MLEs for the parameters are obtained. Alternatively, maximizing the aforementioned equations using any optimization method, such as *optim* in the statistical software R yields the estimators.

5. Simulation

This section outlines three simulation study cases aimed at evaluating the performance of the ML estimates for the parameters of the DAPWE distribution. Various true parameter values were examined, with the following cases:
Case I : $\theta = 0.1, \beta = 5, \alpha = 3, c = 9$.
Case II : $\theta = 1.3, \beta = 1.3, \alpha = 2.1, c = 0.8$.
Case III : $\theta = 0.09, \beta = 4, \alpha = 3.5, c = 7$.
Each simulation was executed with a total of 10,000 iterations (nsim = 10,000). The mean square error (MSE) was utilized to assess the MLE, $\hat{\omega}$, for each parameter. This can be expressed as:

$$MSE(\hat{\omega}) = \frac{\sum_{i=1}^{nsim} (\hat{\omega}_i - \omega_{tr})^2}{nsim}$$

The method of Monte Carlo simulation was employed in this process. Table 2 displays the parameters estimates along with their MSEs.

Table 2. Results for the simulation for the DAPWE estimates and MSE, with different sample sizes for the three cases.

Sample Size	Parameter	First case		Second case		Third case	
		MLE	MSE	MLE	MSE	MLE	MSE
$n = 30$	θ	0.1001583	3.177414e-06	1.161508	0.08597770	0.09011732	4.374959e-06
	β	4.9990808	5.845506e-05	1.403781	0.06519533	3.99292246	2.853079e-04
	α	2.9922006	7.406129e-03	1.9421084	5.25259323	3.49641563	2.057189e-02
	c	8.9799483	4.653185e-02	0.945816	0.07725800	6.98276053	2.656954e-01
$n = 100$	θ	0.1000493	9.539688e-07	1.1875715	0.06517131	0.09006012	1.215366e-06
	β	4.9999064	6.026195e-06	1.3799060	0.04806201	3.99917415	2.407169e-05
	α	3.0001994	1.596485e-04	1.9653759	3.18410142	3.49624582	1.213938e-03
	c	9.0004404	9.603389e-04	0.8791983	0.02714949	6.98551589	1.717626e-02
$n = 200$	θ	0.1000097	4.805996e-07	1.1991885	0.05635460	0.09001163	6.161602e-07
	β	4.9999947	8.961264e-08	1.3720042	0.04148667	3.99978343	7.671131e-06
	α	2.9999986	9.104468e-07	1.9639074	2.49671081	3.49933736	2.868404e-04
	c	8.9999971	4.985113e-06	0.8627653	0.01949143	6.99737643	4.145624e-03
$n = 500$	θ	0.1000097	1.859450e-07	1.2223791	0.03926199	0.09000877	2.365109e-07
	β	4.9999992	8.012236e-11	1.3572180	0.02854648	3.99999515	6.855065e-08
	α	3.0000000	3.907789e-12	1.9421084	1.53148079	3.49999323	3.504064e-07
	c	9.0000001	2.551322e-11	0.8443388	0.01145764	6.99997387	6.656688e-06

From Table 2 and Figures 3, 4 , 5, it is clearly obvious that the estimates become closer to the true value of parameters and the MSE become smaller as the sample size increase.

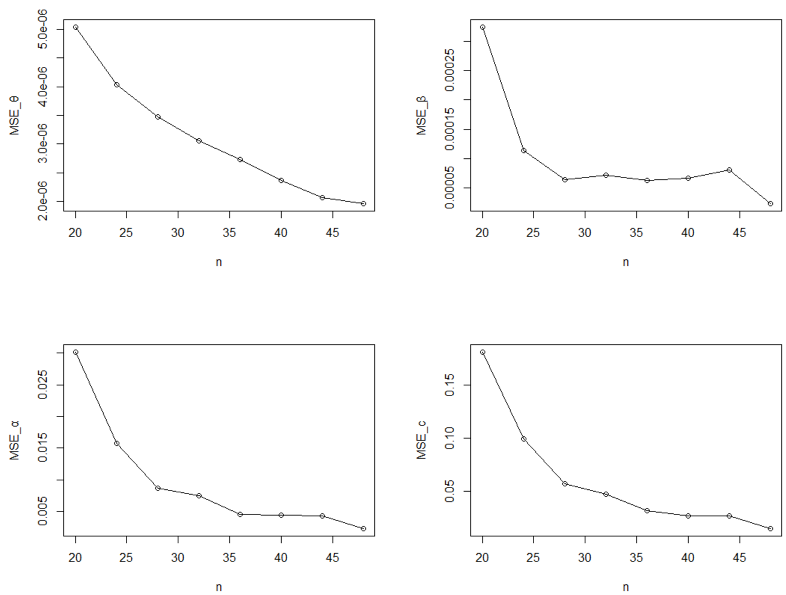

Figure 3. The MSE of the estimates for DAPWE (0.1,5,3,9).

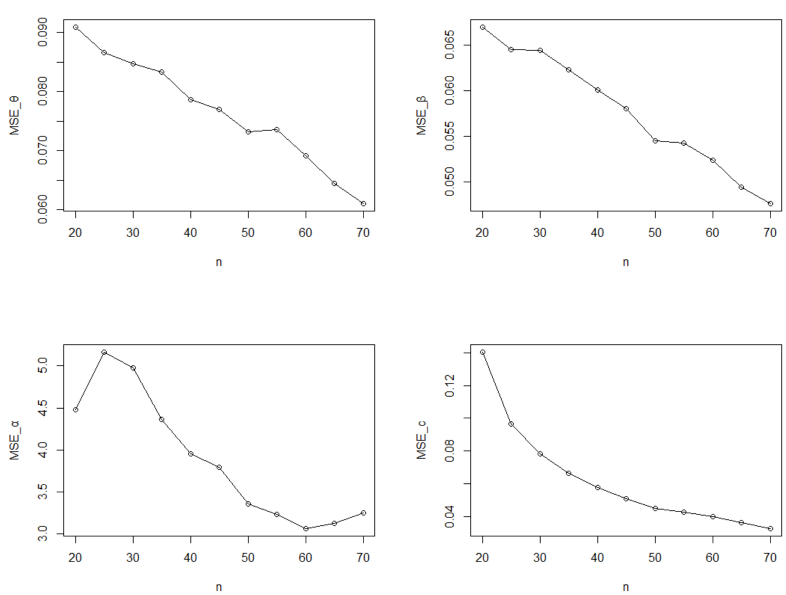

Figure 4. The MSE of the estimates for DAPWE (1.3,1.3,2.1,0.8).

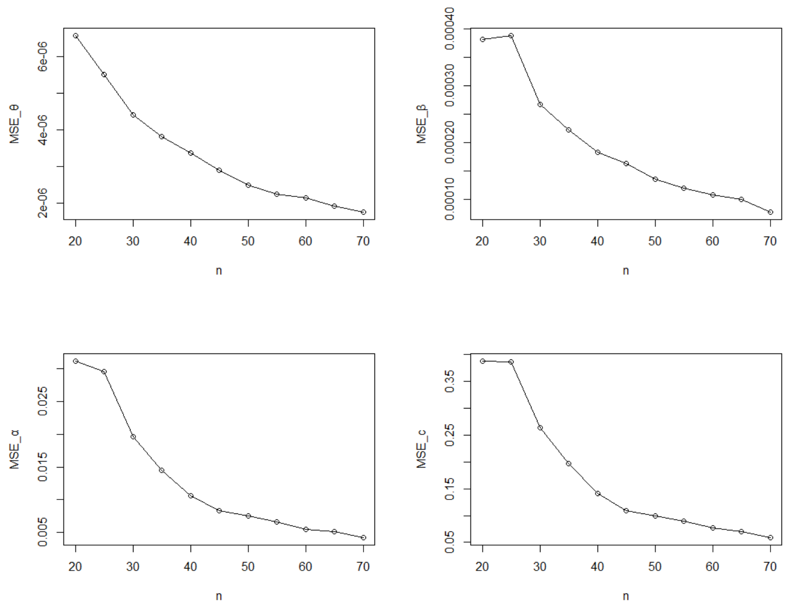

Figure 5. The MSE of the estimates for DAPWE (0.09,4,3.5,7).

6. Applications

Four real data sets in this section were used to be fitted using six various models, including the DAPWE. The PMFs for the competitive models are shown as

(1) Discrete Weibull Exponential (DWE) distribution by [14]

$$f(x;\alpha,\beta,\theta) = \exp\left(-\left(\frac{\theta x}{\beta}\right)^{\alpha}\right) - \exp\left(-\left(\frac{\theta(x+1)}{\beta}\right)^{\alpha}\right),\ \alpha,\beta,\theta > 0.$$

(2) Poisson distribution (Pois) by [22].

$$f(x;\lambda) = \frac{\lambda^{x} e^{-\lambda}}{x!}, \lambda > 0.$$

(3) Discrete Alpha Power Weibull (DAPW) by [23]

$$f(x;\alpha,\beta,\theta) = \frac{\alpha}{\alpha-1}\left[\left(1-\alpha^{-\theta^{x^{\beta}}}\right)-\left(1-\alpha^{-\theta^{(x+1)^{\beta}}}\right)\right], 0 < \theta < 1,\ \alpha,\beta > 0, \alpha \neq 1.$$

(4) Discrete Burr-Hatke (DBH) by [24]

$$f(x;\lambda) = \left(\frac{1}{x+1} - \frac{\lambda}{x+2}\right)\lambda^{x},\ 0 < \lambda < 1.$$

(5) Discrete Weibull Geometric Distribution (DWGeom) by [25]

$$f(x;\rho,p,\alpha) = \frac{(1-p)\left(\rho^{x^{\alpha}} - \rho^{(x+1)^{\alpha}}\right)}{\left(1-p\,\rho^{x^{\alpha}}\right)\left(1-p\,\rho^{(x+1)^{\alpha}}\right)},\ 0 < \rho < 1,\ 0 < p < 1,\ \alpha > 0.$$

(6) Discrete Pareto (DPareto) by [26]

$$f(x;\theta) = \exp\Big(-\theta\log(x+1)\Big) - \exp\Big(-\theta\log(x+2)\Big),\ \theta > 0.$$

To identify the best model, various information criteria (IC) were adopted, including Akaike's Information Criterion (AIC) [27], the Corrected Akaike Information Criterion (AICc) [28], the Hannan-Quinn Information Criterion (HQIC) [29], and the Kolmogorov-Smirnov (K-S) test with its p-value. Additionally, plots were presented to compare the DAPWE distribution's effectiveness with the aforementioned ones.

Data set I:

This data set, reported by [30], represents the daily COVID-19 death toll for Argentina that was tracked over a period of 65 days, from June 1 to August 4, 2020. The data are: 20, 11, 19, 10, 18, 27, 27, 14, 14, 28, 19, 24, 31, 30, 17, 23, 20, 24, 43, 25, 25, 13, 24, 33, 36, 39, 43, 25, 25, 28, 38, 27, 53, 40, 50, 37, 33, 79, 52, 53, 42, 38, 31, 41, 67, 61, 85, 61,71, 42, 35, 145, 80, 111, 105, 125, 66, 43, 126, 118, 111, 155, 77, 69, and 55.

The p-p plot and the empirical CDF for the fitted models are presented in Figures 6 and 7. Table 3 summarizes the results of the MLE's of parameters, including standard errors (SEs) for each distribution, log-likelihood, AIC, AICc, HQIC, and P-Values of K-S.

Table 3. MLE's (and their corresponding SE's in parentheses) for data set I.

Distributions	DAPWE	DWE	Pois	DAPW	DBHatke	DWGeom	DPareto
Paremeters estimation	$\hat{\theta}$=0.0252	$\hat{\theta}$=0.015	$\hat{\lambda}$=48.5695	$\hat{\alpha}$=1.045	$\hat{\lambda}$=0.9997	$\hat{p}$=0.071	$\hat{\theta}$=0.2694
	(0.039)	(0.015)	(0.864)	(0.8986)	(0.002)	(0.384)	(0.033)
	$\hat{\beta}$=2.087	$\hat{\beta}$ = 0.8149	– – –	$\hat{\beta}$=1.55	– – –	$\hat{\rho}$=0.9951	– – –
	(3.283)	(0.8255)	– – –	(0.0491)	– – –	(0.00196)	– – –
	$\hat{\alpha}$=0.0758	$\hat{\alpha}$ = 1.4947	– – –	$\hat{\theta}$=0.998	– – –	$\hat{\alpha}$ = 1.3513	– – –
	(0.1038)	(0.1388)	– – –	(0.00012)	– – –	(0.0799)	– – –
	c=1.8626	– – –	– – –	– – –	– – –	– – –	– – –
	(0.1767)	– – –	– – –	– – –	– – –	– – –	– – –
-logL	306.0943	308.5531	854.5994	308.4608	482.5658	309.5786	391.5444
AICc	620.8552	623.4996	1711.2622	623.3151	967.1951	625.5506	785.1524
AIC	620.1885	623.1062	1711.1987	622.9217	967.1316	625.1571	785.0889
HQIC	623.6203	625.6800	1712.0567	625.4955	967.9896	627.7309	785.9468
P-Value	0.4047033	2.40546e-01	5.232e-12	1.5467e-01	6.799e-48	3.2035e-01	7.185e-14

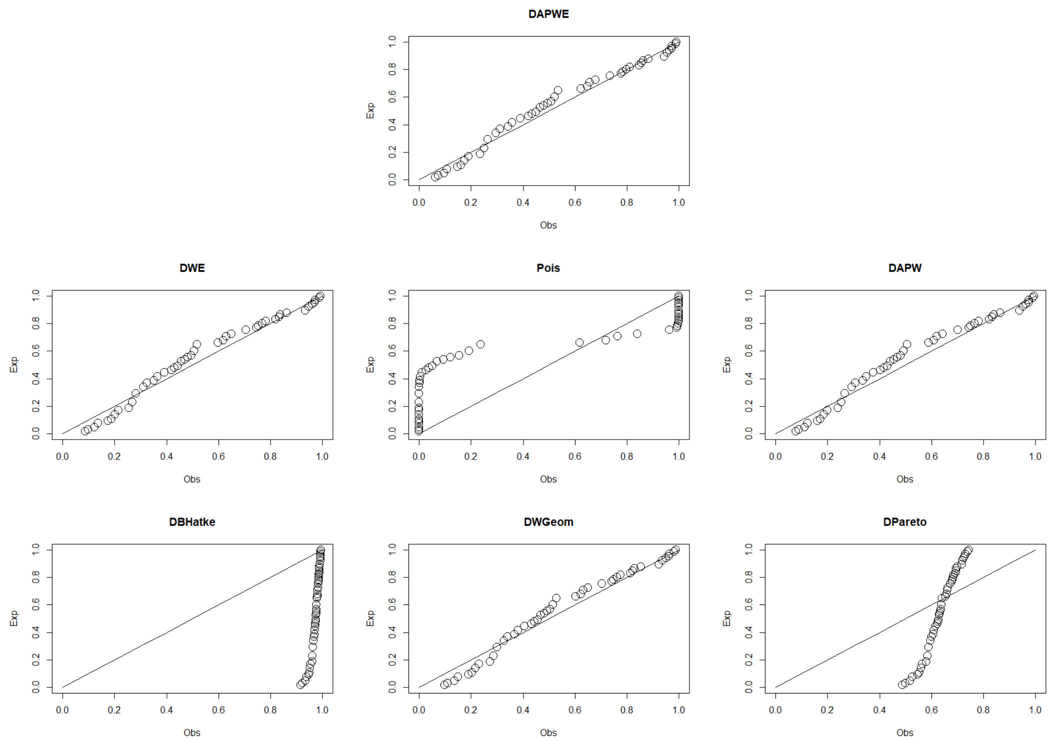

FIGURE 6. The p-p plots for data set I.

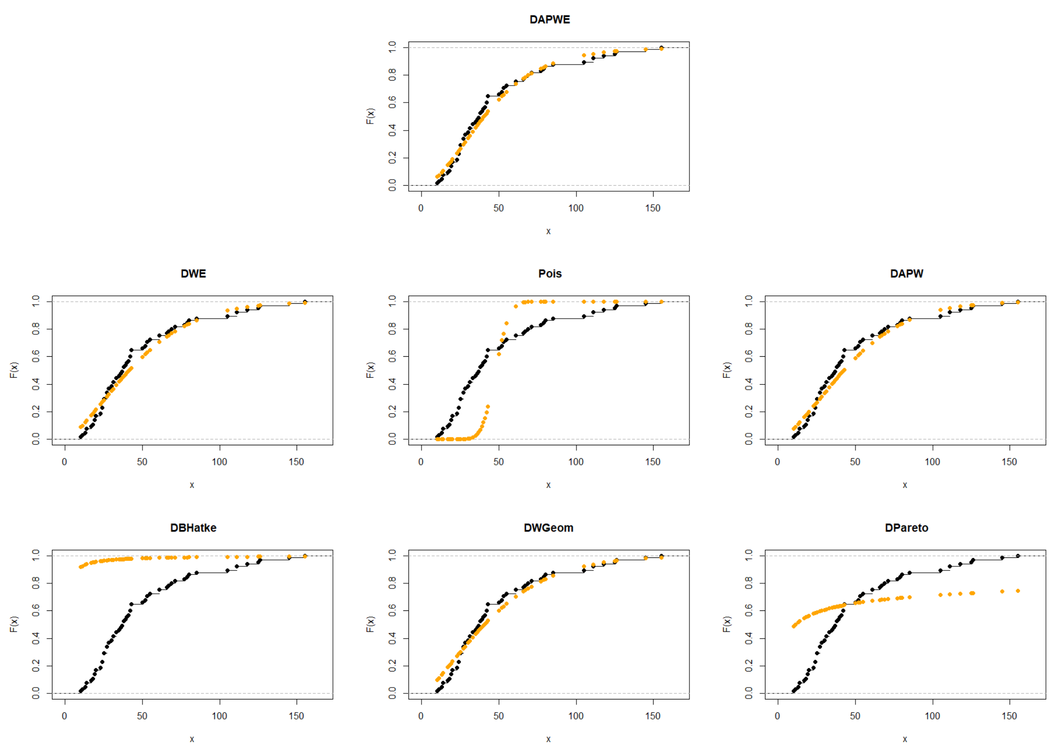

FIGURE 7. The empirical CDFs plots for data set I.

Data set II:

This data set was presented by [23] and includes the mathematics degrees of 48 students from the Indian Institute of Technology in Kanpur. The data are as follows: 29,25, 50 ,15, 13 ,27, 15, 18, 7,7 ,8 ,19, 12 ,18, 5 ,21, 15, 86, 21 ,15 ,14 ,39 ,15, 14, 70, 44, 6, 23, 58, 19, 50 ,23, 11, 6, 34, 18, 28 ,34, 12, 37, 4, 60, 20, 23 ,40 ,65, 19, 31.

The p-p plot and the empirical CDF for the fitted models are presented in Figures 8 and 9. Table 4 summarizes the results of the MLE's of parameters, including standard errors (SEs) for each distribution, log-likelihood, AIC, AICc, HQIC, and P-Values of K-S.

Table 4. MLE's (and their corresponding SE's in parentheses) for data set II.

Distributions	DAPWE	DWE	Pois	DAPW	DBHatke	DWGeom	DPareto
Paremeters estimation	$\hat{\theta}$=0.0327	$\hat{\theta}$=0.00479	$\hat{\lambda}$=25.896	$\hat{\alpha}$=3.923	$\hat{\lambda}$=0.999	$\hat{p}$=0.1673	$\hat{\theta}$=0.3225
	((0.0764)	(0.0025)	(0.7345)	(5.9395)	(0.0046)	(0.4479)	(0.0465)
	$\hat{\beta}$=1.4585	$\hat{\beta}$=0.12129	− − −	$\hat{\beta}$=1.278	− − −	$\hat{\rho}$=0.991	− − −
	(3.4374)	(0.067)	− − −	(0.2125)	− − −	(0.00725)	− − −
	$\hat{\alpha}$=0.081	$\hat{\alpha}$=0.83187	− − −	$\hat{\theta}$=0.9808	− − −	$\hat{\alpha}$=1.3754	− − −
	(0.1423)	(0.1059)	− − −	(0.0209)	− − −	(0.16655)	− − −
	c=1.8327	− − −	− − −	− − −	− − −	− − −	− − −
	(0.2064)	− − −	− − −	− − −	− − −	− − −	− − −
-logL	197.4232	210.6193	396.5892	199.2364	297.6761	199.3776	251.1808
AICc	403.7767	427.7841	795.2654	405.0183	597.4392	405.3006	504.4486
AIC	402.8465	427.2387	795.1784	404.4728	597.3523	404.7551	504.3617
HQIC	405.675	429.3601	795.8856	406.5942	598.0594	406.8765	505.0688
P-Value	0.8982	0.001827	4.3416e-07	6.6073e-01	< 2.2e-16	0.623357	9.4026e-09

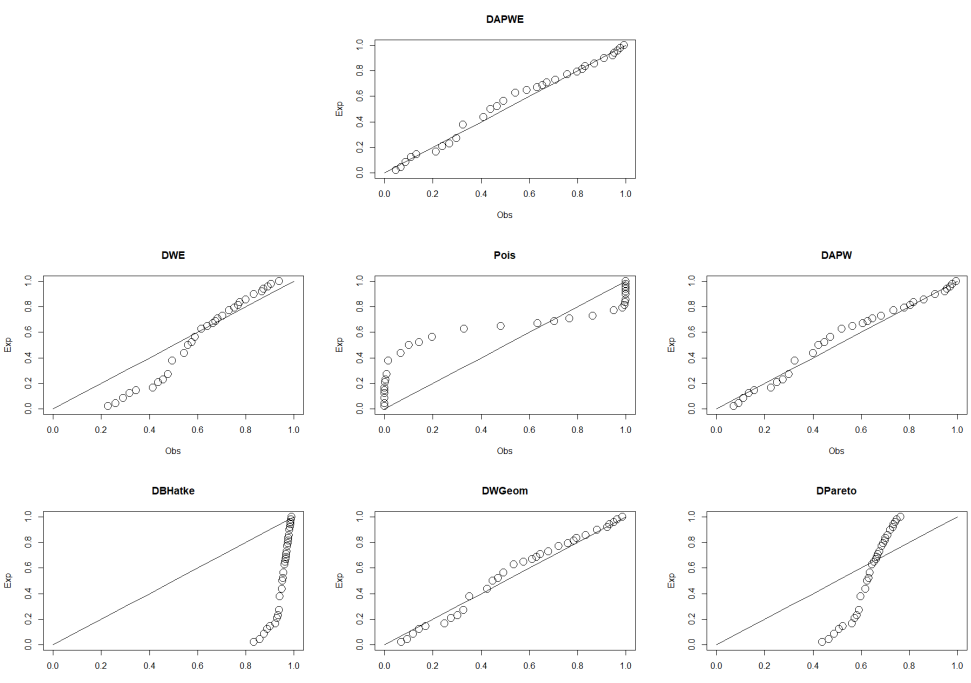

Figure 8. The p-p plots for data set II.

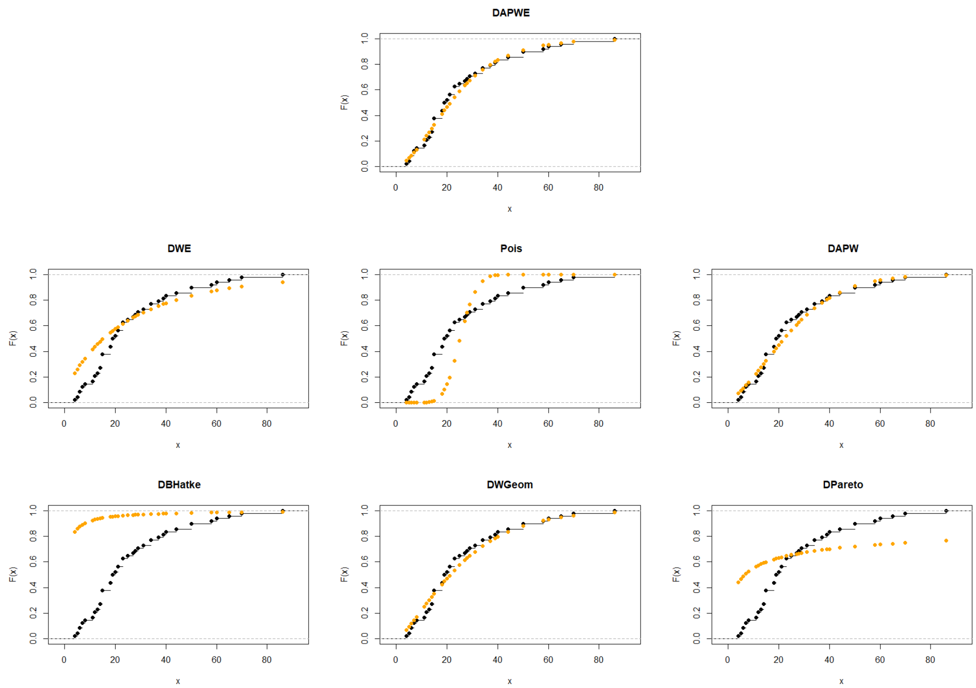

FIGURE 9. The empirical CDFs plots for data set II.

Data set III:

This data set was submitted by [30], it presents the daily new COVID-19 cases for Angola, documented over 27 days from July 8 to August 3, 2020. The data are: 33, 10, 62, 4, 21, 23, 19, 16, 35, 31, 31, 49, 18, 44, 30, 33, 39, 29, 36, 16, 18, 50, 78, 31, 39, 16, 116.

The p-p plot and the empirical CDF for the fitted models are shown in Figures 10 and 11. Table 5 summarizes the results of the MLE's of parameters, including standard errors (SEs) for each distribution, log-likelihood, AIC, AICc, HQIC, and P-Values of K-S.

TABLE 5. MLE's (and their corresponding SE's in parentheses) for data set III.

Distributions	DAPWE	DWE	Pois	DAPW	DBHatke	DWGeom	DPareto
Paremeters estimation	$\hat{\theta}$=0.0307	$\hat{\theta}$=0.0183	$\hat{\lambda}$=34.333	$\hat{\alpha}$=2.018	$\hat{\lambda}$=0.9994	$\hat{p}$=0.1406	$\hat{\theta}$=0.2935
	(0.08)	(0.0334)	(1.1277)	(2.7604)	(0.0047)	(0.4748)	(0.0565)
	$\hat{\beta}$=2.1032	$\hat{\beta}$=0.7156	– – –	$\hat{\beta}$=1.5411	– – –	$\hat{\rho}$=0.9935	– – –
	(5.5202)	(1.3077)	– – –	(0.1201)	– – –	(0.005)	– – –
	$\hat{\alpha}$=0.0177	$\hat{\alpha}$=1.66157	– – –	$\hat{\theta}$=0.9957	– – –	$\hat{\alpha}$= 1.3693	– – –
	(0.0431)	(0.2301)	– – –	(0.0026)	– – –	(0.1629)	– – –
	c=2.0924	– – –	– – –	– – –	– – –	– – –	– – –
	(0.3035)	– – –	– – –	– – –	– – –	– – –	– – –
-logL	116.2471	117.6509	238.7173	117.7497	183.9725	118.8870	152.0907
AICc	242.3124	242.3452	479.5945	242.5428	370.1051	244.8175	306.3414
AIC	240.4942	241.3017	479.4345	241.4993	369.9451	243.7740	306.1814
HQIC	242.0355	242.4577	479.8198	242.6553	370.3304	244.9300	306.5668
P-Value	0.8262	0.598906	3.4e-03	5.9801e-01	< 2.2e-16	0.2023379	3.0848e-06

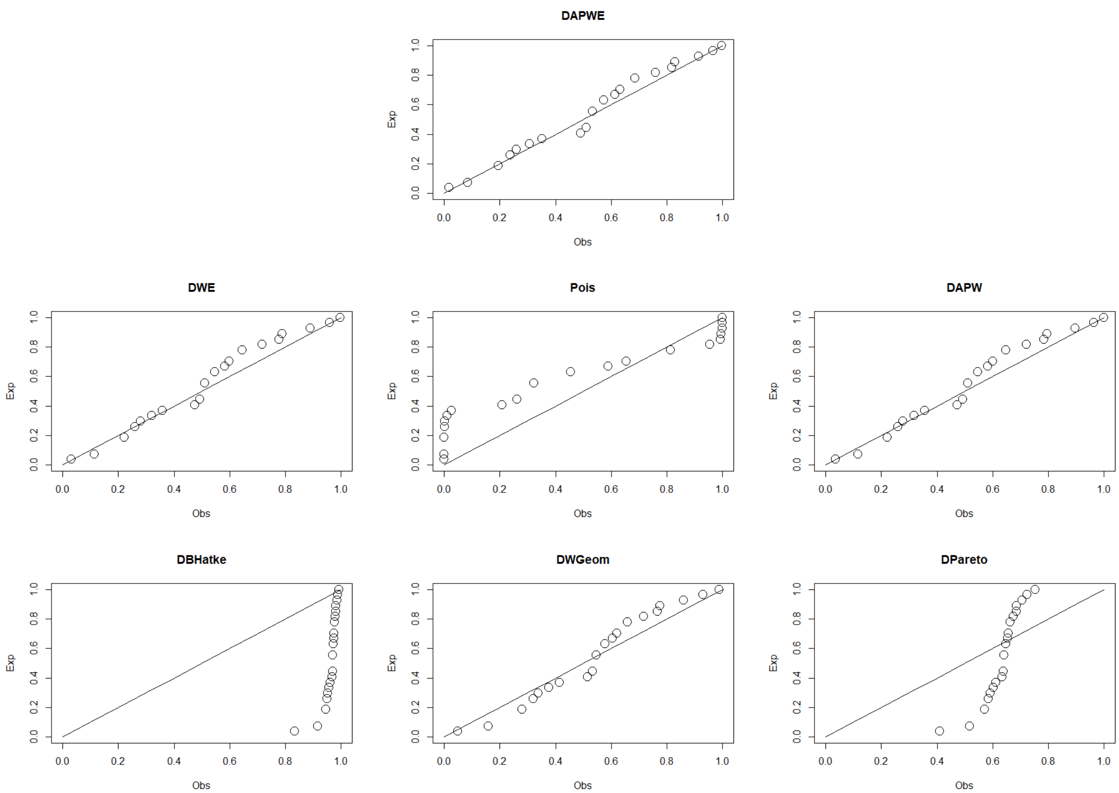

FIGURE 10. The p-p plots for data set III.

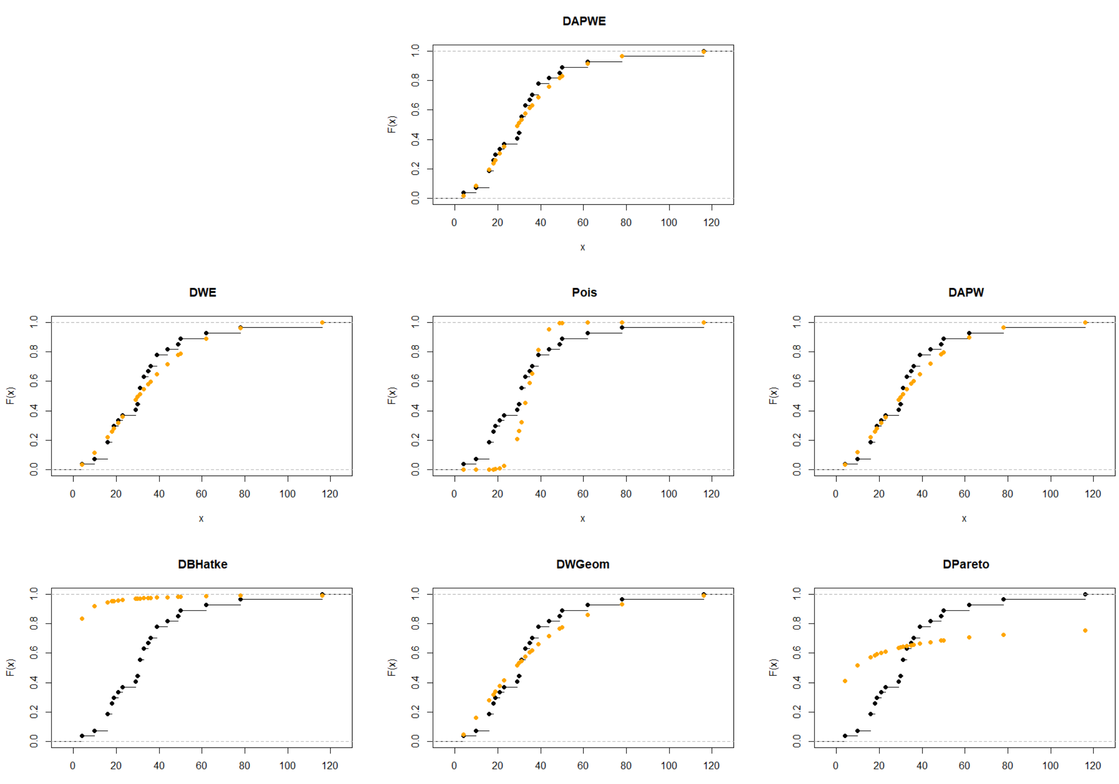

FIGURE 11. The empirical CDFs plots for data set III.

Data set IV:

This data set is from the survival package of R program [31], The data set is about Survival time in 23 patients with Acute Myelogenous Leukemia. The data are as follows: 9 , 13 , 13 ,18 , 23 ,28 , 31 ,34 , 45 ,48, 161 , 5, 5 , 8, 8 , 12 , 16, 23, 27 ,30 ,33, 43 , 45.

The p-p plot and the empirical CDF for the fitted models are shown in Figures 12 and 13. Table 6 summarizes the results of the MLE's of parameters, including standard errors (SEs) for each distribution, log-likelihood, AIC, AICc, HQIC, and P-Values of K-S.

Table 6. MLE's (and their corresponding SE's in parentheses) for data set IV.

Distributions	DAPWE	DWE	Pois	DAPW	DBHatke	DWGeom	DPareto
Paremeters estimation	$\hat{\theta}$=0.0276	$\hat{\theta}$=0.0232	$\hat{\lambda}$=29.478	$\hat{\alpha}$=1.04496	$\hat{\lambda}$=0.9994	$\hat{p}$=0.0024	$\hat{\theta}$=0.3194
	(0.0872)	(0.0698)	(1.132)	(0.8986)	(0.00497)	(1.175)	(0.0666)
	$\hat{\beta}$=2.093	$\hat{\beta}$=0.748	− − −	$\hat{\beta}$=1.553	− − −	$\hat{\rho}$=0.9843	− − −
	(6.6367)	(2.24993)	− − −	(0.0491)	− − −	(0.02485)	− − −
	$\hat{\alpha}$=0.0077	$\hat{\alpha}$=1.19888	− − −	$\hat{\theta}$=0.998	− − −	$\hat{\alpha}$= 1.195	− − −
	(0.01704)	(0.1724)	− − −	(0.000117)	− − −	(0.230)	− − −
	c=1.5649	− − −	− − −	− − −	− − −	− − −	− − −
	(0.24012)	− − −	− − −	− − −	− − −	− − −	− − −
-logL	98.52762	100.49461	302.69560	100.57088	144.00921	100.49622	121.26128
AICc	207.2775	208.2524	607.5817	208.4049	290.2089	208.2556	244.7130
AIC	205.0552	206.9892	607.3912	207.1418	290.0184	206.9924	244.5226
HQIC	206.1975	207.8459	607.6768	207.9985	290.3040	207.8492	244.8081
P-Value	0.9855025	6.995549e-01	0.0006324	7.381453e-01	4.049202e-15	0.63614484	1.04793e-04

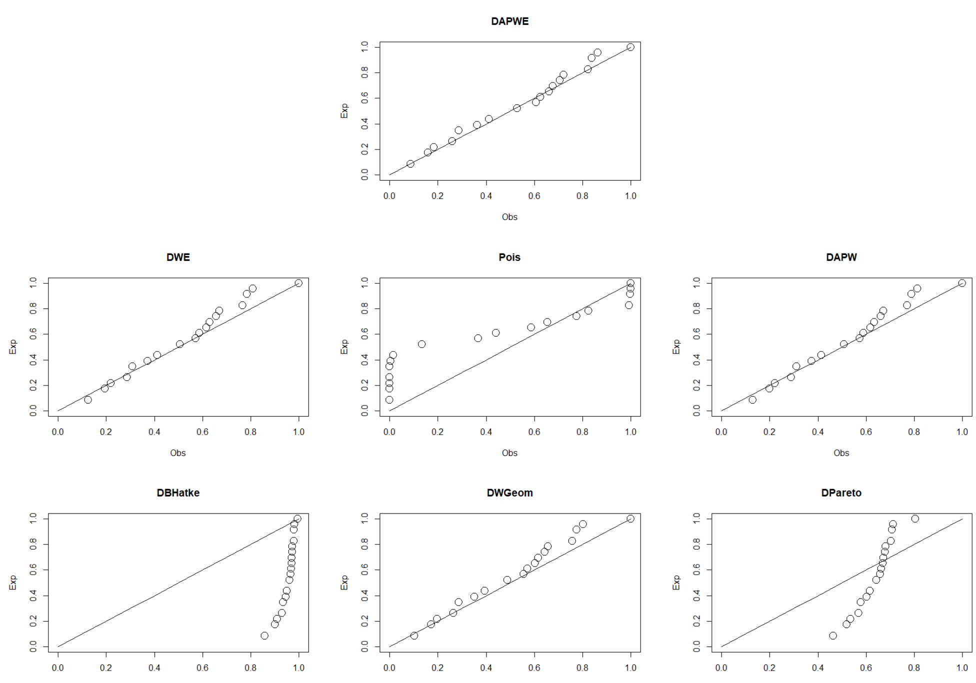

Figure 12. The p-p plots for data set IV.

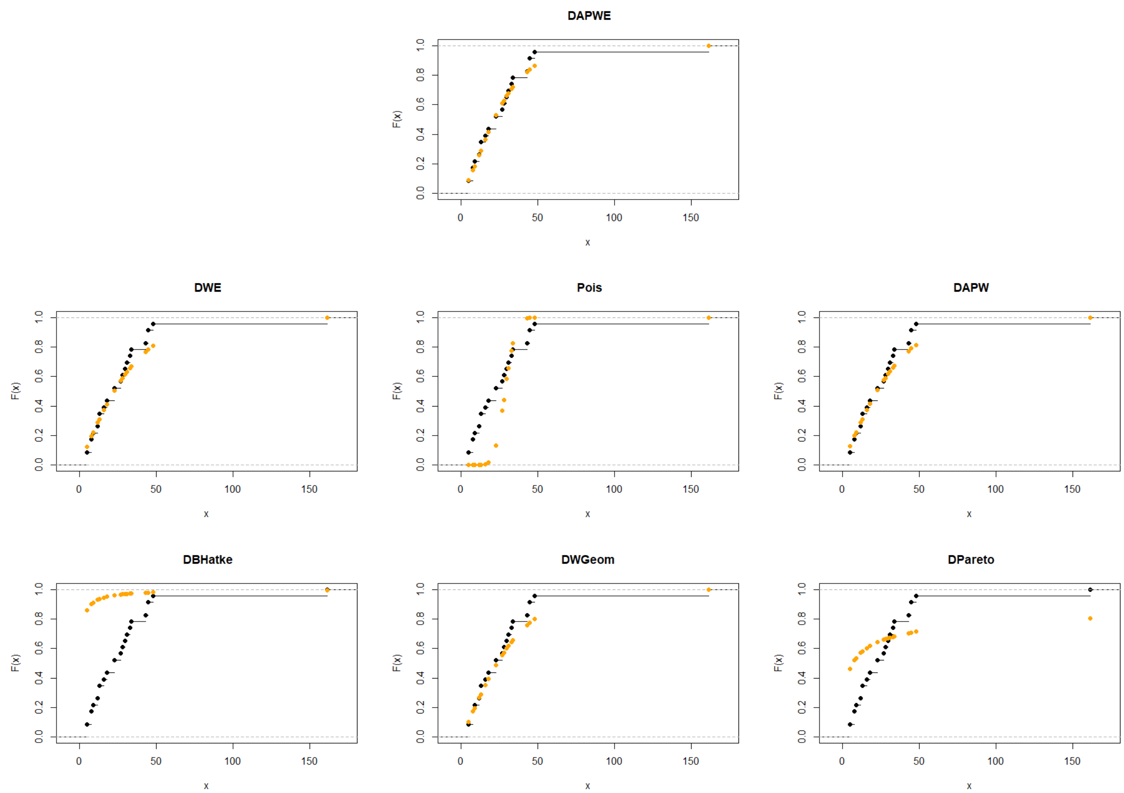

FIGURE 13. The empirical CDFs plots for data set IV.

The tables 3, 4, 5 and 6 clearly indicate that the DAPWE distribution achieves superior efficiency compared to competing distributions, as evidenced by the AIC, AICc, HQIC, and P-Values of K-S. Additionally, this conclusion is further supported by Figures 6, 8, 10, 12 and Figures 7, 9, 11 and 13.

7. Conclusion

This paper introduces a discretization method for a continuous family of distributions, referred to as the Discrete Alpha Power Weibull-G family. Within this framework, the four-parameter DAPWE distribution is introduced. Its PMF and HRF exhibit attractive forms for adopting different data behaviors. Some statistical characteristics were analyzed, and the method of maximum likelihood was employed to estimate its four parameters. To evaluate the performance of the (MLEs), three scenarios with varying parameter values of the DAPWE and four various sample sizes were examined. Furthermore, to demonstrate the effectiveness of the DAPWE distribution compared to other distributions, four real data sets were utilized. The findings highlighted the effectiveness and usefulness of the DAPWE, showing that the DAPWE distribution is highly adaptable for modeling real data in different fields.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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