

CHAPTER 41

CURRENT DEPTH REFRACTION USING FINITE ELEMENTS

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ABSTRACT

A computational model is presented for the steady state prediction of currents, waves, and sea surface elevations in a coastal region outside the surf zone. Irrotational flow of surface gravity waves on large-scale steady currents over a gently sloping sea bed is considered. A second order Stokian approach is used, and all dynamic quantities are integrated over depth and averaged over time, in that order. The flow equations and the boundary conditions are presented. A method is developed for the solution of the non-linear steady model by introducing a sequence of two-level calculations, viz. a 'wave level' and a 'current level'. The variables are split on the two levels. The wave field is found, using that the flux of wave action is constant between adjacent wave rays. The current field and the mean sea surface elevation are determined using a Galerkin finite element method. The current field is approximated by triangular elements with linear interpolation functions, and the mean sea surface elevation is approximated by a triangular element with quadratic (parabolic) interpolation functions. A quasi-two-dimensional test solution is tabulated.

1. INTRODUCTION

The purpose of the present computational model is to make a steady state prediction of currents, waves, and elevations of the mean sea surface in a coastal or offshore region outside the surf zone, when the topography of the gently sloping sea bed is given, see Fig. 1.

The computation of depth refraction of regular surface gravity waves presents no fundamental difficulties, see e.g. Skovgaard et al. (1975, 1976) and Skovgaard and Petersen (1976). When the wave height H is small, the combined effect of diffraction and depth refraction can also be calculated in the shallow water region, see Berkhoff (1973, 1975), Chen and Mei (1974, 1976), and Zienkiewicz and Bettess (1975, 1976). On greater water depths, where the intermediate depth wave theory must be applied, combined diffraction and refraction has only been determined for a very simple geometry of the sea bed, see Jonsson et al. (1976b). In the presence of a current the problem becomes much more complex on account of current-wave interaction and anisotropy. Therefore we exclude reflection and diffraction and consider only combined current

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depth refraction, i.e. the approach is within the framework of geometrical optics.

Subcritical, irrotational flow of large-scale, horizontally non-uniform and vertically uniform, steady currents over an arbitrary gently sloping sea bed is considered. This means that variations in current velocity and depth are slow, in the sense that changes are appreciable only over many wave lengths. Surface and bottom shear are disregarded, and so is dissipation. The waves are periodic and progressive with the constant absolute wave period T_a ($= 2\pi/\omega_a$), ω_a being the absolute angular frequency. Stokes' second order wave theory is used, and the wave fronts are long-crested. Fluxes of mass, momentum, and energy are integrated over depth and averaged over T_a , in that order. The two-dimensional flow region Ω is simply connected with waves in the whole domain. The fluid is incompressible and homogeneous with density ρ . The Coriolis force and the spherical form of our planet are neglected. The non-linear steady model will be solved by a sequence of two-level calculations, viz. a 'wave level' and a 'current level'.

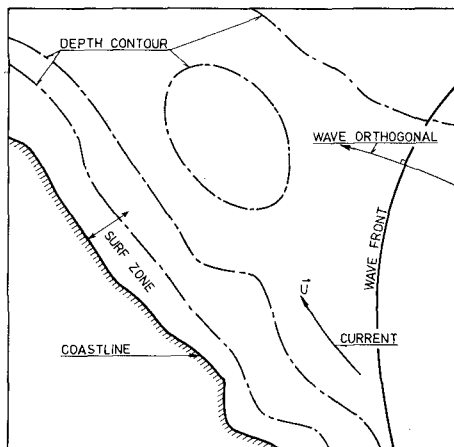


Fig. 1

Horizontal sketch of a coastal region

Stokes' second order wave theory is used, and the wave fronts are long-crested. Fluxes of mass, momentum, and energy are integrated over depth and averaged over T_a , in that order. The two-dimensional flow region Ω is simply connected with waves in the whole domain. The fluid is incompressible and homogeneous with density ρ . The Coriolis force and the spherical form of our planet are neglected. The non-linear steady model will be solved by a sequence of two-level calculations, viz. a 'wave level' and a 'current level'.

Another model for current wave interaction in two horizontal dimensions, but with bed shear, is reported by Noda et al. (1974) and Noda (1976). This model is like the present one solved by a sequence of two level iterations; however, the approaches on both calculation levels are completely different from our.

The governing equations for irrotational current depth refraction were introduced by Jonsson (1971b), see also Jonsson and Wang (1976). These equations were solved in one horizontal dimension (no refraction) by Jonsson et al. (1971). Plane flow with vorticity is being reported, see Brink-Kjær (1976), Jonsson et al. (1976a), and Jonsson (1977).

The effect of a combined current wave field is in principle that, when the current has a positive component in the direction of wave propagation, the waves are 'lengthened' and so 'feel bottom' at a greater depth than without the current. So in this case the bending of the wave orthogonal in the combined field is stronger than for pure waves. Conversely, a negative current component will 'shorten' the waves, and the bending of the wave orthogonal is less than for no current (see later).

2. THE GOVERNING EQUATIONS

The arbitrary bottom topography is known (measured from a horizontal datum) in a fixed horizontal Cartesian coordinate system (x,y) . It was

later), (5) conservation of zero vorticity, (6) conservation of wave crests, and (7) conservation of bottom topography. The conservation equations given by Jonsson (1971b) become in the above mentioned order

$$\frac{\partial}{\partial x}(h u) + \frac{\partial}{\partial y}(h v) = 0 \quad (1)$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{E}{2} (G + (1+G) \cos^2 A) + \rho h u^2 \right) - \rho g h \frac{\partial \eta}{\partial x} \\ + \frac{\partial}{\partial y} \left(\frac{E}{2} (1+G) \cos A \sin A + \rho h u v \right) = 0 \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{E}{2} (1+G) \cos A \sin A + \rho h u v \right) - \rho g h \frac{\partial \eta}{\partial y} \\ + \frac{\partial}{\partial y} \left(\frac{E}{2} (G + (1+G) \sin^2 A) + \rho h v^2 \right) = 0 \end{aligned} \quad (3)$$

$$\frac{\partial}{\partial x} \left(\frac{E}{\omega_r} (u + c_{gr} \cos A) \right) + \frac{\partial}{\partial y} \left(\frac{E}{\omega_r} (v + c_{gr} \sin A) \right) = 0 \quad (4)$$

$$\frac{\partial}{\partial x} \left(v - \frac{E \sin A}{\rho h c_r} \right) - \frac{\partial}{\partial y} \left(u - \frac{E \cos A}{\rho h c_r} \right) = 0 \quad (5)$$

$$c_a = c_r + u \cos A + v \sin A \quad \text{or} \quad \omega_a = \omega_r + k(u \cos A + v \sin A) \quad (6a,b)$$

$$D = h + \eta \quad (7)$$

where we have introduced 5 new (secondary) unknowns: The absolute phase speed (see Fig. 3)

$$c_a = L/T_a \quad (\equiv \omega_a/k) \quad (8)$$

L being the wave length, the relative phase and group speeds

$$c_r = \sqrt{\frac{g}{k} \tanh kh} \quad (9) \quad c_{gr} = \frac{1}{2} c_r (1+G) \quad (10)$$

the relative angular frequency ω_r and the parameter G

$$\omega_r = c_r k \quad (11) \quad G = 2kh / \sinh 2kh \quad (12)$$

In the shallow water limit we have $G = 1$ and $c_r = c_{gr} = \sqrt{gh}$. If we insert these limiting values in eqs. 1-7, we get the shallow water version of the conservation equations, see Skovgaard and Jonsson (1976b).

As shown by Skovgaard and Jonsson (1976a) it is possible to replace any of the conservation equations with the differential equation for the wave orthogonal or with the algebraic equation for the set-down (see later), or to replace any two of the conservation equations with these two equations.

The differential equation for the wave orthogonal reads, see e.g. Skovgaard and Jonsson (1976a)

$$\frac{DA}{Ds} = \frac{1}{c_a} \left(\sin A \frac{\partial c_a}{\partial x} - \cos A \frac{\partial c_a}{\partial y} \right) \quad (13)$$

where s is a coordinate along the wave orthogonal (positive in the direction of the wave propagation). Together with eq. 13 we have the two parametric differential equations

$$Dx/Ds = \cos A \quad (14) \quad Dy/Ds = \sin A \quad (15)$$

The equation for the current wave set-down reads, see Jonsson (1971a, 1976)

$$\eta = \frac{1}{2g}(u^2 + v^2) + \frac{E}{\rho g h} \left(\frac{G}{2} - (u \cos A + v \sin A)/c_r \right) \quad (16)$$

The direction of the wave ray is given by the direction of the absolute group velocity vector $\vec{c}_{ga} (= \vec{U} + \vec{c}_{gr})$, and so is determined by

$$\tan \mu = \frac{v + c_{gr} \sin A}{u + c_{gr} \cos A} \quad (17)$$

where μ is the angle from the x-axis to $\vec{c}_{ga}$, see Fig. 3. When eq. 17 is differentiated with respect to a coordinate r along the wave ray, the differential equation for the wave ray is obtained

$$\frac{D\mu}{Dr} = \cos^2 \mu \frac{D}{Dr} \left(\frac{v + c_{gr} \sin A}{u + c_{gr} \cos A} \right) \quad (18)$$

Together with the wave ray equation we have two parametric equations

$$Dx/Dr = \cos \mu \quad (19) \quad Dy/Dr = \sin \mu \quad (20)$$

Remark that the wave ray equation presented by Jonsson (1971b), eq. 30, is not correct.

3. BOUNDARY CONDITIONS

The current vector $\vec{U}$ must be given everywhere on the boundary Γ which surrounds Ω , see Fig. 4. The boundary is divided in two parts, Γ_1 and Γ_2 . Γ_1 is that part of Γ where all rays are running in (or out) of Ω , see Fig. 4, and along it the wave field must be known. (Along Γ_2 the wave field is determined by the calculations.)

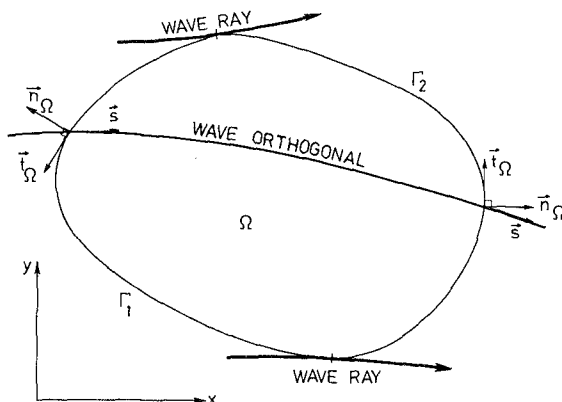


Fig. 4
Definition sketch
of flow domain Ω
with the boundary
 $\Gamma = \Gamma_1 + \Gamma_2$

The wave field is in principle defined by T_a , A , E , L , and h ; however, in an irrotational model it is not necessary to prescribe h in more than one point of Γ_1 for the following reason. For given $\vec{U}$, T_a , A , E , and h in one point of Γ_1 , we can calculate η in that point by

iteration in eqs. 6, 8, 9, 12, and 16, and then D using eq. 7. Thereby the MISL is determined, which is horizontal in Ω . Combining eqs. 6, 7, 8, 9, 12, and 16 we then can calculate h for all other points of Γ_1 .

We shall later show that it is more convenient if the (incoming) waves are prescribed by the curvature of the ray front κ_e and by A (or μ) in some selected points along Γ_1 rather than by the continuous variation of A along Γ_1 . The angles A and μ are connected by eq. 17 in any point where $\vec{U}$ and h are known, i.e. if A is given we can determine μ , and vice versa. The curvature of the wave action front κ_e is determined by a pure geometrical consideration

$$\kappa_e \equiv D\mu/De = -\sin\mu \partial\mu/\partial x + \cos\mu \partial\mu/\partial y = (D(De)/Dr)/De \quad (21)$$

where De is the infinitesimal distance between two rays, see Fig. 6. Since r is not constant along e -lines, and vice versa, lengths r and e do not form a set of curvilinear coordinates. This is the reason for using the formalism D/Dr instead of $\partial/\partial r$, etc.

In summary we have

$$\vec{U} = \vec{U}_{\Gamma_1} \quad \text{along } \Gamma_1 \quad (22a)$$

$$\kappa_e = \kappa_{e,\Gamma_1} \quad E = E_{\Gamma_1} \quad A = A_{\Gamma_1} \quad \text{or} \quad \mu = \mu_{\Gamma_1} \quad (22b)$$

in some selected points of Γ_1

$$h = h_1 \quad \text{in one point of } \Gamma_1 \quad (22c)$$

$$\vec{U} = \vec{U}_{\Gamma_2} \quad \text{along } \Gamma_2 \quad (22d)$$

For the present irrotational steady state model we can make two integral checks of volume fluxes and velocities. The normal component of $\vec{q}$ ($=h\vec{U}$) along Γ ($=\Gamma_1 + \Gamma_2$) is controlled by mass conservation

$$\int_{\Gamma} \vec{q} \cdot \vec{n}_{\Omega} d\Gamma = 0 \quad (23)$$

where $\vec{n}_{\Omega}$ is a unit vector normal to the boundary of and going outward from Ω , and we have assumed that Ω is without sources or sinks. As h is not known along Γ_2 (η is a function also of the unknown wave field) we cannot check in advance that eq. 23 is exactly fulfilled. However, we can calculate h along Γ corresponding to a pure current, and it is expected that for a 'realistic' problem the error introduced in eq. 23 by neglecting the wave contribution to h is small, since it is a second order term. The wave field could also be estimated along Γ_2 before using eq. 23. One could formulate the above as: Eq. 23 must be roughly fulfilled in advance for a realistic problem.

The tangential component of the mean velocity is controlled by the condition of zero vorticity. Introducing the Eulerian mean velocity vector $\vec{U}_t$ ($\equiv \vec{U} - E/(\rho h c_r) \vec{s}$) below wave trough level ($\vec{s}$ being a unit vector in the direction of wave travel), Stokes' theorem gives

$$\int_{\Omega} \text{rot } \vec{U}_t d\Omega = \int_{\Gamma} \vec{U}_t \cdot \vec{t}_{\Omega} d\Gamma \quad (24)$$

where $\vec{t}_{\Omega}$ is a unit vector tangential to Γ going counterclockwise around Ω . Using eq. 5 we therefore get

$$\int_{\Gamma} \left(\vec{U} - \frac{E}{\rho h c_r} \vec{s} \right) \cdot \vec{t}_{\Omega} d\Gamma = 0 \quad (25)$$

The wave contribution to the integrand in eq. 25 is a second order term which furthermore has not the same sign along Γ . Thus it can be inferred that the advance neglect of the wave term in eq. 25 will correspond to a realistic problem. Also here we can say: Eq. 25 must be roughly fulfilled in advance for a realistic problem.

4. THE TWO-LEVEL APPROACH

The current depth refraction model is solved by a two-level iterative scheme. In this way we so to speak 'split' the unknowns in two groups, and we do not have to solve all equations simultaneously. The two levels will be named the wave level and the current level.

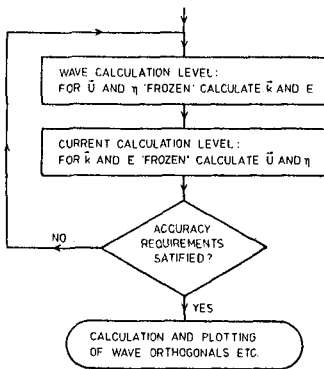


Fig. 5 The two-level splitting of the calculations

In the wave level calculations the 'propagation medium' is held fixed ('frozen'), i.e. $\vec{U}$, h , and η are known in Ω on this level, and $\vec{k}$ and E are the primary unknowns, see Fig. 5.

In the current level calculations the wave field is frozen, i.e. $\vec{k}$ and E are known in Ω on this level, and $\vec{U}$, h , and η are the primary unknowns. The governing equations on this level are non-linear, which implies that we have to iterate during the calculations.

The calculations are initiated on a wave level, where we assume $\vec{U} = 0$ and $\eta = 0$ in Ω . The two-level iteration scheme is continued until the differences between

two consecutive levels of the same type are lower than a chosen limit for both the wave and the current level, i.e. the number of iterations is determined by accuracy requirements.

After the completion of the two-level iterations the paths of some wave orthogonals are calculated by numerical integration of the three ordinary differential equations 13 - 15, and the wave field is plotted with the wave heights and the current components written at discrete points along these orthogonals. Also the phase field, i.e. the wave fronts is calculated and plotted after the completion of the two-level iterations. Eqs. 13 - 15 cannot be used in the given form to calculate the phase field. Time t must be used as independent variable in this case

$$dA/dt = \sin A \partial c_a / \partial x - \cos A \partial c_a / \partial y \quad (13a)$$

$$dx/dt = c_a \cos A \quad (14a)$$

$$dy/dt = c_a \sin A \quad (15a)$$

5. THE WAVE LEVEL

The approach on this level is a generalization of the method which is used in depth refraction calculations, see e.g. Skovgaard et al. (1975).

The wave field is determined from one-point conditions (initial conditions) by integration along characteristic lines. By a suitable placing of the boundary points (initial points) along Γ_1 , Ω is covered by a set of characteristic lines. The method is attractive, because we only have to solve a set of four ordinary differential equations along each of the lines.

In depth refraction the characteristic lines are the wave orthogonals, and in current depth refraction the characteristic lines are the wave rays which, of course, for $U = 0$ coincide with the wave orthogonals. The path of the wave ray is determined by eqs. 18-20. Bretherton and Garrett (1968) have shown generally that there is no flux of wave action E/ω_r across a wave ray, i.e. between two rays we have a constant flux of wave action, see eq. 4. For water waves, Jonsson (1971a, 1976) found the same result, using quite a different approach. For $U = 0$ the wave action is, of course, proportional with wave energy density, the flux of which is constant between two orthogonals.

$$De_{st} \left(\frac{E}{\omega_r} c_{ga} \right)_{st} = De \frac{E}{\omega_r} c_{ga}$$

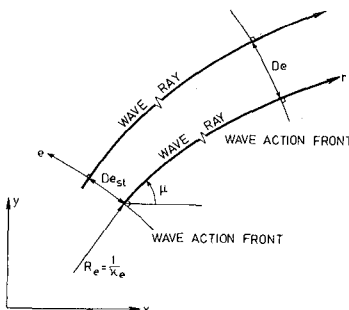


Fig. 6 Two adjacent wave rays and two 'wave action fronts'

Munk and Arthur (1952) derived a differential equation for the wave *orthogonal* separation factor β for $U = 0$. In almost the same manner it is possible to derive a differential equation for the wave *ray* separation factor β_r with distance r along the ray as the independent variable. The factor β_r is defined by

$$\beta_r \equiv De/De_{st} \quad \beta_r > 0 \quad (26)$$

in which suffix 'st' denotes a value at the starting or initial point along Γ_1 , see Fig. 6. The separation factor is found in the following way. Introducing eq. 26 in eq. 21 we get

$$D\mu/De = \beta_r^{-1} D\beta_r/Dr \quad (27)$$

Using the operators

$$\frac{D}{Dr} = \cos \mu \frac{\partial}{\partial x} + \sin \mu \frac{\partial}{\partial y} \quad (28)$$

$$\frac{D}{De} = -\sin \mu \frac{\partial}{\partial x} + \cos \mu \frac{\partial}{\partial y} \quad (29)$$

we have

$$\frac{D}{De} \left(\frac{D\mu}{Dr} \right) - \frac{D}{Dr} \left(\frac{D\mu}{De} \right) = \left(\frac{D\mu}{Dr} \right)^2 + \left(\frac{D\mu}{De} \right)^2 \quad (30)$$

(or $D\kappa_r/Ds - D\kappa_e/Dr = \kappa_r^2 + \kappa_e^2$, κ_r being the curvature of the wave ray). Inserting eq. 27 and using eqs. 28-29 once more, we get an ordinary second-order homogeneous differential equation for β_r

$$\frac{D^2 \beta_r}{Dr^2} + q_s \beta_r = 0 \quad (31)$$

$$q_s = \left[\left(\frac{\partial \mu}{\partial x} \right)^2 - \left(\frac{\partial \mu}{\partial y} \right)^2 - \frac{\partial^2 \mu}{\partial x \partial y} \right] \cos 2\mu + \left[\frac{1}{2} \left(\frac{\partial^2 \mu}{\partial x^2} - \frac{\partial^2 \mu}{\partial y^2} \right) + 2 \frac{\partial \mu}{\partial x} \frac{\partial \mu}{\partial y} \right] \sin 2\mu \quad (32)$$

The coefficient q_s depends on some explicitly known quantities (u , v , h , and η , and their partial derivatives up to the second order) and on some implicitly known quantities (A and E , etc., and their partial derivatives up to second order). The implicit quantities are calculated from some of the conservation equations and the wave orthogonal equation 13.

The condition of constant flux of wave action between two adjacent rays determines in principle E (and so the wave height)

$$E = \frac{E_{st}}{\beta_r} \frac{\omega_r}{\omega_{r,st}} \frac{c_{ga,st}}{c_{ga}} \quad (33)$$

where we have used eq. 26, and c_{ga} ($\equiv |\vec{c}_{ga}|$) is the absolute group speed.

The associated one-point boundary conditions at $r = r_{st}$ for the four ordinary differential equations 18-20 and 31 are

$$\mu = \mu_{st} \quad x = x_{st} \quad y = y_{st} \quad \beta_r = \beta_{r,st} \quad \frac{D\beta_r}{Dr} = \beta_{r,st} \kappa_{e,st} \quad (34)$$

where x_{st} and y_{st} are chosen on Γ_1 , μ_{st} is calculated from eq. 17, $\beta_{r,st}$ is arbitrarily chosen as 1, and $\kappa_{e,st}$ is the curvature of the wave action front in the considered point of Γ_1 . When the differential equation for β_r (eq. 31) is written as a system of two simultaneous first-order equations, eqs. 18-20 and 31 constitute a system of five simultaneous first-order ordinary differential equations, where the five requisite 'initial' conditions at $r = r_{st}$ are given by eq. 34. The system is integrated step-by-step along one ray at a time. A standard variable order code with automatic local control of the error level and corresponding adjustable step length is used.

The calculation along a wave ray stops when one of the following conditions is fulfilled: (1) The boundary of the region of analysis is reached, (2) the orthogonals converge too much, (3) the Stokes parameter is too high (thereby we in practice exclude wave breaking), (4) the bottom slope is too high, (5) the water depth is too small, or (6) the rays converge too much.

The first five conditions equal those for depth refraction, see Skovgaard et al. (1975). Here condition (2) was formulated $D\beta/Ds < -\beta/(1L)$, where l probably is about one. However, since $\beta^{-1} D\beta/Ds = DA/Df = -\sin A \partial A/\partial x + \cos A \partial A/\partial y$, we need not solve the differential equation for β to test condition (2), (Df is the infinitesimal distance between two orthogonals). Nor do we have to determine the wave orthogonal paths. Condition (6) is of the same type as condition (2). Condition (3) reads $HL^2/h^3 > 20$ (say), thus excluding non-Stokian waves.

The solution (i.e. $\vec{k}$, E etc. and the associate position (x,y) of the ray) is recorded in the computer at a set of prescribed values of distance r along the path of the wave ray. Note that the paths of the rays are new on each level, since the propagation properties of the medium changes on the current level.

6. THE CURRENT LEVEL

On this level the wave field (i.e. $\vec{k}$ and E) is frozen in Ω , and $\vec{U}$, h and η are the primary unknowns. Using eq. 7, and remembering that D is also known in Ω , we immediately can eliminate h , which implies that we have only three scalar unknowns u , v and η on this level. These three variables are determined by three conservation equations, viz. conservation of mass, eq. 1, and conservation of momentum in the two horizontal directions, eqs. 2-3. We solve the partial differential equations by a direct finite element method, the so-called Galerkin weighted residual process (i.e. a weak formulation). The region Ω is approximated with a domain $\bar{\Omega}$ (with boundary $\bar{\Gamma}$) consisting of a finite number N of triangular subdomains ω . The only difference between Ω and $\bar{\Omega}$ is along the boundary. The triangular subdomains are constructed by connecting the prescribed points along the wave rays with straight lines, see Fig. 7. As the rays are new on each wave level, the triangular subdomains are also new on each current level.

It is well known that we cannot freely select the polynomial order of the shape functions (interpolation functions), see e.g. Hood and Taylor (1974) and Olson and Tuann (1976). We choose to apply the most simple shape functions, which for the present differential equations 1-3 are a linear shape function for u and v , and a quadratic shape function for η . Remark the simple connection in eq. 16 between u , v and η .

It should be explained why we choose to eliminate the physical depth h and not the set-down η . If η was eliminated, and a shape function for h chosen, the indirectly assumed form of the MWS would be influenced by the local form of the sea bed (see eq. 7 and Fig. 2).

We select the most simple placing of the nodal points, i.e. we use for the linear interpolation functions the three vertices as nodal points (see Fig. 8(a)), and for the quadratic interpolation functions the three vertices plus the three mid-side points as nodal points, see Fig. 8(b). Implicit in the preceding discussion is the fact that the same triangular subdomains are used for set-down and velocity fields. Details of these standard elements can be found in e.g. Zienkiewicz (1971), chapter 7. With these two elements the u , v and η solutions become continuous but not differentiable along the sides of the triangular domains. In a typical domain ω we have the following approximations

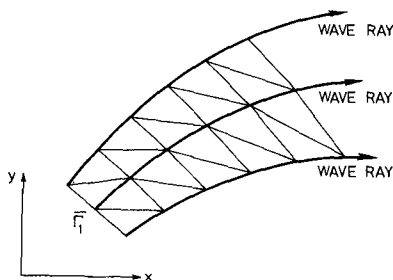
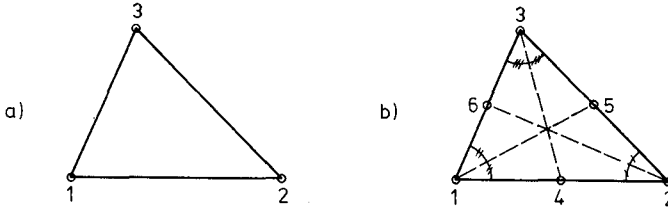


Fig. 7 Placing of triangular elements

$$u = \psi_j(x,y) \bar{u}_j \quad v = \psi_j(x,y) \bar{v}_j \quad \eta = \phi_l(x,y) \bar{\eta}_l \quad (35a,b,c)$$

where the bar indicates a nodal value, and ψ_j , $j = 1, 2, 3$, and ϕ_l , $l = 1, 2, \dots, 6$ are the shape functions. Here and henceforward the repeated nodal indices j , l , etc. are summed over the range of the index.



o NODAL POINT

Fig. 8 Placing of nodal points in isoparametric triangular elements
(a) linear three-node element, (b) quadratic six-node element

For the conservation equation of mass (where h is eliminated) we form the residual $r_e(x,y)$ at point $(x,y) \in \omega$ by inserting eqs. 35 a-c in the left hand side of eq. 1

$$r_e(x,y) = \bar{u}_j \frac{\partial}{\partial x} (D\psi_j) - \bar{\eta}_l \bar{u}_j \frac{\partial}{\partial x} (\phi_l \psi_j) + \bar{v}_j \frac{\partial}{\partial y} (D\psi_j) - \bar{\eta}_l \bar{v}_j \frac{\partial}{\partial y} (\phi_l \psi_j) \quad (36)$$

The residual r_e is multiplied with weight functions which for this residual are the shape functions for the set-down ϕ_m ($m = 1, 2, \dots, 6$). Support for the appropriate selection of the weight function is given by e.g. Chung and Chion (1976). The functions $(r_e \phi_m)$, $m = 1, 2, \dots, 6$ are integrated over ω , and each of the results is equated with zero. Oden and Wellford (1972) p. 1592 have given the reason for this approach, 'We can guarantee that the residual vanishes in an average sense over the element by requiring that it be orthogonal with respect to the inner product $\langle f, g \rangle \equiv \int_{\omega} f g d\omega$ to the subspace spanned by the functions $\phi_m(x,y)$, $m = 1, 2, \dots, 6$. Then $\langle r_e, \phi_m \rangle = \int_{\omega} r_e \phi_m d\omega = 0$, and we obtain the finite element model of the mass conservation equation.'

Using $\int_{\omega} r_e \phi_m d\omega = 0$, we get

$$\bar{u}_j A_{mj} - \bar{\eta}_l \bar{u}_j B_{mlj} + \bar{v}_j C_{mj} - \bar{\eta}_l \bar{v}_j D_{mlj} = 0 \quad m = 1, 2, \dots, 6 \quad (37)$$

and A_{mj} , B_{mlj} , C_{mj} , and D_{mlj} denote the four local arrays

$$\left. \begin{aligned} A_{mj} &\equiv \int_{\omega} \phi_m \frac{\partial}{\partial x} (D\psi_j) d\omega & B_{mlj} &\equiv \int_{\omega} \phi_m \frac{\partial}{\partial x} (\phi_l \psi_j) d\omega \\ C_{mj} &\equiv \int_{\omega} \phi_m \frac{\partial}{\partial y} (D\psi_j) d\omega & D_{mlj} &\equiv \int_{\omega} \phi_m \frac{\partial}{\partial y} (\phi_l \psi_j) d\omega \end{aligned} \right\} \quad (38)$$

In the same manner we obtain the finite element model of the two momentum conservation equations, the only difference being that the weight functions are ψ_m , $m = 1, 2, 3$ for both equations. For eq. 2 we get

$$\begin{aligned}
& \rho \bar{u}_j \bar{u}_p E_{mjp} - \rho \bar{\eta}_1 \bar{u}_j \bar{u}_p F_{mljp} - \rho g \bar{\eta}_1 G_{ml} + \rho g \bar{\eta}_1 \bar{\eta}_n H_{mln} \\
& + \rho \bar{u}_j \bar{v}_p I_{mjp} - \rho \bar{\eta}_1 \bar{u}_j \bar{v}_p J_{mljp} \\
& = - \int_{\omega} \psi_m \left(\frac{\partial}{\partial x} \left(\frac{E}{2} (G + (1+G) \cos^2 A) \right) + \frac{\partial}{\partial y} \left(\frac{E}{2} (1+G) \cos A \sin A \right) \right) d\omega \quad (39)
\end{aligned}$$

$m = 1, 2, 3$; where E_{mjp} , F_{mljp} , G_{ml} , H_{mln} , I_{mjp} , and J_{mljp} denote 6 local arrays, which are not presented. Nor are the similar equations for eq. 3.

Off-node function values and derivatives of u , v , h , and η , which are used in the wave level calculations, are not formed directly from the finite element method (FEM) solution, using the FEM interpolation functions etc. We follow, however, the guidance given by Carl de Borr (1974), see Roache (1975), p. 235, 'The acknowledged best procedure is to ignore the basis function and evaluate off-node function values and derivatives by standard interpolation formulas.'

Eqs. 37 and 39, and the similar one for eq. 3 represent a set of 12 (= 6 + 3 + 3) non-linear algebraic equations in 12 unknowns, which are $\bar{\eta}_1$, $l = 1, 2, \dots, 6$, $\bar{u}_j$, $j = 1, 2, 3$, and $\bar{v}_j$, $j = 1, 2, 3$. Upon assembling the elements, the global stiffness matrix form is sparse, but the system of algebraic equations is non-linear. Along the boundary $\bar{\Gamma}$, the specified current components $\bar{u}$ and $\bar{v}$ are inserted directly in the global matrix form. Using these specified values of $\bar{u}$ and $\bar{v}$ we can calculate $\bar{\eta}$ along $\bar{\Gamma}$ from eq. 16 by iteration, using that E , A , c_r , and G are known from the calculations on the wave level. These values of $\bar{\eta}$ are then inserted directly into the global matrix form.

The global non-linear system of algebraic equations is solved using the Newton Raphson iteration scheme, see e.g. Dahlquist and Björck (1974), p. 250. The iteration is started by interpolation in the current level solution used on the preceding wave level. The number of iterations is determined by accuracy requirements.

Combined diffraction refraction FEM models for short waves are rather expensive in use, as the minimum number of elements is a function of the wave length. The present model for current depth refraction does not have this restriction. The number of elements in our model is a function of the depth variations and the associated variations of the current field. Only the variation of the wave height (amplitude envelope) is modelled. In contrast a diffraction refraction FEM model describes simultaneous amplitudes, i.e. rapid oscillations in horizontal space.

It should be explained why we choose to use both momentum equations, and do not substitute one of them with eq. 16. If we had done this, our model would have been 'tailored' to irrotational flow only, and a later inclusion of bottom friction would be less straightforward. With the present approach we can instead use eq. 16 as a check equation in the nodal points.

7. TEST SOLUTION

For straight and parallel (but arbitrarily spaced) sea bed contours, with the current $\bar{u}$ everywhere parallel with these, a test solution is

calculated numerically (assuming plane incidence in deep water). With these constraints we have a quasi two-dimensional problem in that no phase-independent parameters vary with distance along-shore. Therefore we can calculate the solution (except the ray and orthogonal paths) with D as the only independent variable, without integrating the differential equations for the rays and the orthogonals. In order to simplify the presentation we place the x -axis at right angles to the contours, $\vec{U} = (0, v)$, or $U = v$, and $\delta = \pm 90^\circ$.

For the present problem we can directly integrate the conservation equations (4 and 5) for wave action and zero vorticity

$$\frac{\omega}{\epsilon_r} E c_{gr} \cos A = \text{const}_1 \quad (40)$$

$$v - \frac{E \sin A}{\rho h c_r} = \text{const}_2 \quad (41)$$

When the y -component of the conservation (3) for momentum is integrated and combined with eqs. 8, 10, 11, and 40 we get Snell's law (see Jonsson, 1971a, 1976)

$$c_a / \sin A = \text{const}_3 \quad (42)$$

Further we can easily derive algebraic formulas for the ray and orthogonal separation factors β_r and β ($\equiv Df/Df_{st}$)

$$\beta_r = \cos \mu / \cos \mu_{st} \quad (43)$$

$$\beta = \cos A / \cos A_{st} \quad (44)$$

Note that by combining eqs. 40 and 43 we get eq. 33.

The solution, i.e. the 12 unknowns ($k, A, E, v, h, n, c_r, c_a, \omega_r, c_{gr}, G$, and μ) as a function of D , is found by numerical iteration in 12 non-linear algebraic equations (Nos. 6, 7, 8, 9, 10, 11, 12, 16, 17, 40, 41, and 42). After the completion of the iteration, β_r and β are found from eqs. 43 and 44. The ray path is deter-

General information		g	A_{st}^0	δ^0	T_a	D_{st}	$(D\beta_r/Dr)_{st}$	$v_h \rightarrow \infty$	$\beta_{r,st}$	H_{st}
SI units		9.80665	60	90	8	20	(-4)4.0230	1	1	1
x	y	r	A^0	μ^0	L	D	$\eta = D - h$	v	β_r	H
(+2)2.1117	(+2)3.7262	0	(+1)6.0000	(+1)6.3224	(+1)9.8709	(+1)2.0000	(-2)5.2248	1.0046	1.0000	1.0000
(+2)4.0274	(+2)6.4111	(+2)4.2848	(+1)5.3510	(+1)5.7385	(+1)9.1635	(+1)1.5777	(-2)5.2682	1.0050	1.1964	(-1)9.1866
(+2)6.1355	(+2)8.7224	(+2)7.5846	(+1)4.6666	(+1)5.1315	(+1)8.2905	(+1)1.1945	(-2)5.3411	1.0059	1.3874	(-1)8.6927
(+2)7.2377	(+2)9.6903	(+3)1.0715	(+1)3.7557	(+1)4.3540	(+1)6.9477	7.7290	(-2)5.5334	1.0087	1.6091	(-1)8.5039
		(+3)1.2183	(+1)3.1720	(+1)3.8907	(+1)5.9927	5.5245	(-2)5.7931	1.0127	1.7273	(-1)8.6628

TABLE 1 Test solution, plane sloping sea bed, slope 1:50

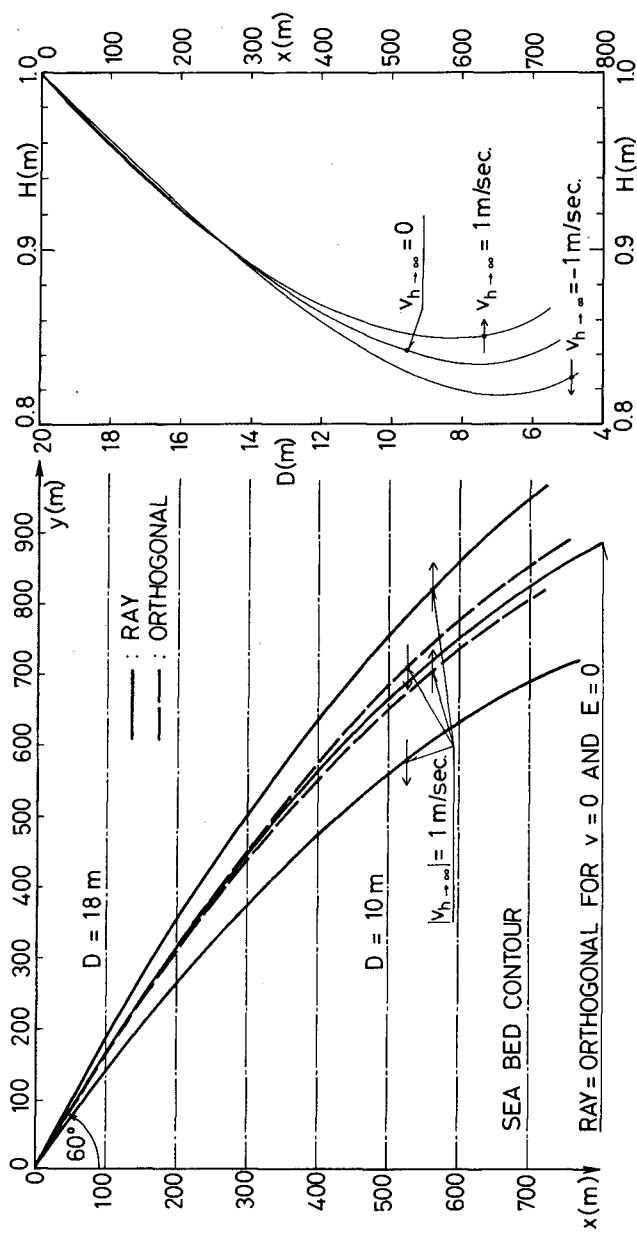


Fig. 9 Paths of wave orthogonals and rays, and variation of wave height.
The effect of a 'positive' and a 'negative' current.
At $D_{st} = 20$ m the angle A_{st} is 60° . Plane sloping sea bed, slope 1:50

mined by simultaneous integration of the two parametric differential equations for the ray, eqs. 19 and 20 (independent variable r). Similarly the orthogonal path is found by another simultaneous integration of eqs. 14-15 (independent variable s).

The wave level part of the general program can be separately checked using the given test solution.

For a plane sloping sea bed (slope 1:50), $T_a = 8$ sec, $H_{st} = 1$ m, $A_{st} = 60^\circ$, $v_h \rightarrow \infty = \pm 1$ m/sec, i.e. $v_{st} = +1.0046$ m/sec, and $v_{st} = -0.99497$ m/sec (both values are rounded with 5 significant digits, 5S), and $D_{st} = 20$ m, the paths of the rays and the orthogonals, and the variation of the wave height are given in Fig. 9. Note that in both cases shown, the rays turn with the current, in contrast to the orthogonals. This phenomenon is explained in the introduction. The figure gives also the path of the orthogonal for $v_{st} = 0$ (and $E_{st} = 0$, i.e. the current $O(H^2)$ is also vanishing, so $v = 0$). It can be shown that if we are in shallow water, the influence of the current on the orthogonal and on the wave height is exceedingly small in this case.

In Table 1 one of the test solutions in Fig. 9 is tabulated for some points along the wave ray. Integers in parentheses indicate powers of 10 by which the following numbers are to be multiplied.

8. CONCLUSION

The irrotational flow equations and the boundary conditions for current depth refraction of surface gravity waves on large-scale steady currents are presented for second-order Stokes waves. An iterative solution algorithm is formulated using a two-level splitting of the calculations, viz. a wave calculation level and a current calculation level. On the wave level, the wave rays are determined in a 'frozen' medium, and wave heights are found using that the flux of wave action is constant between neighbouring rays. On the current level, the waves are 'frozen', and the current field and elevation of the mean surface are determined from the conservation equations for mass and momentum using that the mean irrotational stagnation level (MISL) is horizontal. These conservation equations are solved by a direct finite element method, using triangular linear finite elements for the current and triangular quadratic finite elements for the elevation of the mean sea surface. A test solution is presented, and tabulated for one set of the parameters. The described model is currently being implemented in a general current depth refraction program.

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APPENDIX A: Programming

The numerical methods described are programmed in the IBM OS 360/370 implementation of PL/I. All the floating point calculations are made with 14 hexadecimal digits, i.e. with about 15 decimal digits. The presented results are calculated with about 10S, and the numbers in the test solution in Table 1 are rounded to 5S.

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