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MINIMIZERS FOR FRACTIONAL SCHRÖDINGER EQUATIONS WITH INHOMOGENEOUS PERTURBATION

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Abstract. In this article, we study a constrained minimization problem arising in fractional Schrodinger equations with inhomogeneous term $m(x) \not\equiv 1$. We obtain the existence and limit behavior of constraint minimizers. The argument relies on energy estimates, blow-up analysis, comparison principle and iteration methods.

1. Introduction

Consider the following constraint minimizers of L^2-subcritical fractional variational problem

$$I(M) := \inf_{u\in H^s(\mathbb{R}^N),\, \|u\|_2^2=M} E(u), \tag{1.1}$$

the energy functional $E(u)$ is defined by

$$E(u) := \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx - \frac{2}{p+1}\int_{\mathbb{R}^N} m(x)|u|^{p+1}dx, \tag{1.2}$$

where $N \geq 2$, $s \in (\frac{1}{2}, 1)$, $p \in (1, 1+\frac{4s}{N})$, $M > 0$ and the inhomogeneous term $m(x) \not\equiv 1$ satisfies the assumptions

- (A1) $m(x) \in L^\infty_{\rm loc}(\mathbb{R}^N) \cap C^{1,\alpha}(0 < \alpha < 1)$, $0 < m(x) \leq m(0) = \max_{x\in\mathbb{R}^N} m(x) = 1$, and $0 < \inf_{x\in\mathbb{R}^N} m(x) = \lim_{|x|\to\infty} m(x) = m_\infty < 1$;
- (A2) $0 \in \mathbb{R}^N$ is the unique global maximum point of $m(x)$, and $1 - m(x) = |x|^{r+2}(1 + o(1))$ as $|x| \to 0$, where $r > 0$.

It is well known that the fractional Laplacian $(-\Delta)^s (s \in (0, 1))$ can be defined by

$$(-\Delta)^s v(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x-y|^{N+2s}}dy = C_{N,s} \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{v(x) - v(y)}{|x-y|^{N+2s}}dy$$

for $v \in S(\mathbb{R}^N)$, where $P.V.$ denotes a Cauchy principal value, $S(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying C^∞ function, $B_\varepsilon(x)$ denotes an open ball of radius ε centered at x and the normalization constant $C_{N,s} = \big(\int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}}\big)^{-1}$, see [5, 31, 37] and the references therein for more details. There are applications of operator $(-\Delta)^s$ in some areas such as fractional quantum mechanics, physics and chemistry, obstacle problems, optimization and finance, conformal geometry and minimal surfaces, see [1, 4, 19, 20, 28, 32] and the references therein for more details.

When $s = 1$, this problem is related to the orbital stability waves in nonlinear Schrödinger equations, which was proposed by Lions in [21]. After the pioneer work of Lions, much attention has been devoted to the study of the Schrödinger equation. Recently, many scholars have studied and extended the well-known Bose-Einstein condensates and time-independent Gross-Pitaevskii equation, the reader is referred to [6, 7, 13, 14, 15, 16, 17, 25, 26]. These works mainly studied the situation when $m(x) \equiv 1$.

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When $s \in (0,1)$, problem (1.1) originated from the fractional nonlinear Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N,$$

where $N \geq 2$, $V : \mathbb{R}^N \to \mathbb{R}$ is an external function and $f(x,u)$ is a nonlinearity. In recent years, the study on equations with the fractional Laplacian has been attracted much interest [2, 9, 10, 11, 23, 33, 38]. Cheng [2] considered the following fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = |u|^{p-1}u, \quad x \in \mathbb{R}^N,$$

where $V(x)$ is an unbounded potential and $1 < p < 1+\frac{4s}{N}$, they proved the existence of ground state by Lagrange multiplier method. Moreover, in [9], Dipierro, Palatucci and Valdinoci obtained the existence and symmetry results for solutions with $V(x) \equiv 1$. Felmer, Quaas and Tan [11] studied the same equation with a more general nonlinearity $f(x,u)$ instead of $|u|^{p-1}u$, they obtained the existence of positive solutions and analysed the regularity and symmetry properties of these solutions.

Du, Tian, Wang and Zhang [10] studied the stationary (i.e., time-independent) fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = \mu u + af(u), \quad x \in \mathbb{R}^N,$$

where $N \geq 2$, $V : \mathbb{R}^N \to \mathbb{R}$ is a trapping potential, $\mu \in \mathbb{R}$ and $a > 0$ are parameters, and f is a subcritical nonlinearity. They proved that the optimal embedding constant for the fractional Gagliardo-Nirenberg-Sobolev inequality can be expressed by exact form, and established the existence, nonexistence and mass concentration of L^2-normalized solutions for the above equation. In addition, under a certain type of trapping potentials, by using some delicate energy estimates, the authors presented a detailed analysis of the concentration behavior of L^2-normalized solutions in the mass critical case.

We note that, when $V(x) = 0$ there is no result on existence and mass concentration behavior for inhomogeneous mass subcritical fractional problems. We are going to study the existence results of minimizers as $M \to \infty$. Before describing more details, let us introduce the following fractional Gagliardo-Nirenberg-Sobolev inequality, see [10].

Lemma 1.1. *Let $p \in (1, 1+\frac{4s}{N})$. Then*

$$\int_{\mathbb{R}^N} |u|^{p+1}dx \leq C_{\rm opt}\Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx)\Big)^{\frac{N(p-1)}{4s}} \Big(\int_{\mathbb{R}^N} |u|^2 dx)\Big)^{\frac{2s(p+1)-N(p-1)}{4s}} \tag{1.3}$$

for $u \in H^s(\mathbb{R}^N)\backslash\{0\}$. This equality is attained by a function $Q(x)$ with the following properties:

(i) *$Q(x)$ is radial, positive and strictly decreasing in $|x|$;*

(ii) *$Q(x) \in H^{2s+1}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ and satisfies*

$$\frac{C_1}{1+|x|^{N+2s}} \leq Q(x) \leq \frac{C_2}{1+|x|^{N+2s}} \quad x \in \mathbb{R}^N; \tag{1.4}$$

(iii) *$Q(x)$ is the unique solution of the fraction Schrödinger equation*

$$(-\Delta)^s u + \frac{2s(p+1)-N(p-1)}{N(p-1)}u - \frac{4s}{N(p-1)}u^p = 0; \tag{1.5}$$

(iv)

$$C_{\rm opt} = \frac{p+1}{2\|Q\|_2^{p-1}}. \tag{1.6}$$

According to Lemma 1.1, by a simple calculation, we can observe that

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}Q|^2 dx = \int_{\mathbb{R}^N} |Q|^2 dx = \frac{2}{p+1}\int_{\mathbb{R}^N} |Q|^{p+1}dx. \tag{1.7}$$

We now state our main results.

Theorem 1.2. *If $m(x)$ satisfies* (A1), *then there exists at least one minimizer of $I(M)$ for any $M \in (0,\infty)$.*

To prove Theorem 1.2, we first prove the strict subadditivity inequality of $I(M)$, then applying the fractional Gagliardo-Nirenberg-Sobolev inequality (1.3) to obtain the uniform boundedness of the minimizing sequences. Moreover, by using the concentration-compactness principle [21, 22], the compactness of minimizing sequences can be obtained.

Motivated by studies [13, 15, 25, 27, 36], we are concerned with the limit behavior of minimizers as $M \to \infty$, we have the following result.

Theorem 1.3. *Suppose $m(x)$ satisfies* (A1) *and* (A2). *Let v_k be a nonnegative minimizer of $I(M_k)$. Then, for any sequence $\{M_k\}$ with $M_k \to \infty$ as $k \to \infty$, there exists a subsequence of v_k, still denoted by v_k, such that v_k has a unique maximum point $\bar{z}_k$ and satisfies*

$$\lim_{k\to\infty} \varepsilon_k^{\frac{2s}{p-1}} v_k(\varepsilon_k x + \bar{z}_k) = Q(x) \quad u \in H^s(\mathbb{R}^N), \tag{1.8}$$

where $\lim_{k\to\infty} \bar{z}_k = 0$, $\varepsilon_k := (\frac{M_k}{a^*})^{-\frac{p-1}{4s-N(p-1)}}$, $a^* = \|Q(x)\|_2^2$ and $Q(x)$ is the unique radially symmetric positive solution of (1.5).

Motivated by Maeda [27], we rewrite the constraint variational problem (1.1) into the equivalent form

$$I_M := \inf_{v\in H^s(\mathbb{R}^N),\, \|v\|_2^2=1} E_M(v), \tag{1.9}$$

where

$$E_M(v) := \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v|^2 dx - \frac{2M^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} m(x)|v|^{p+1} dx \quad 1 < p < 1 + \frac{4s}{N}, \tag{1.10}$$

which implies that $v_M := M^{-1/2} u_M$ is a nonnegative minimizer of I_M and $I_M = M^{-1} I(M)$ if and only if u_M is a nonnegative minimizer of $I(M)$. In other words, the proof of Theorem 1.3 can be equivalent to analyzing the limit behavior of minimizers for (1.9) as $M \to \infty$. Up to some necessary scaling of the minimizers, one can obtain the boundedness of minimizers as $M \to \infty$. Another difficult in studying the limit behavior of v_M is to locate the peak of v_M as $M \to \infty$. Inspired by the works in [24, 25, 27], we introduce the following new constraint variational problem

$$\tilde{I}_M := \inf_{v\in H^s(\mathbb{R}^N), \|v\|_2^2=1} \tilde{E}_M(v), \tag{1.11}$$

where $\tilde{E}_M(v)$ is defined by

$$\tilde{E}_M(v) := \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v|^2 dx - \frac{2M^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} |v|^{p+1} dx \quad 1 < p < 1 + \frac{4s}{N}. \tag{1.12}$$

By establishing that $I_M - \tilde{I}_M \to 0$ as $M \to \infty$, one deduces that $\frac{2M^{\frac{p-1}{2}}}{p+1} \int_{\mathbb{R}^N} (1-m(x))|v_M|^{p+1} dx \to 0$ as $M \to \infty$, which is a good way to locate the peak of minimizers.

The rest of this article is organized as follows. Section 2 is devoted to proving Theorem 1.2 on the existence of minimizers for (1.1). While in remaining section, we give the proof of Theorem 1.3. Throughout this paper, we use the following notation:

- The space $H^s(\mathbb{R}^N)$ is equipped with the norm $\|u\|^2 = \int_{\mathbb{R}^N} (|(-\Delta)^{s/2} u|^2 + |u|^2) dx$;
- The norm in $L^p(\mathbb{R}^N)$ is denoted by $\|\cdot\|_p$, where $p \in [1, \infty]$;
- $C, C_1, C_2, \dots,$ denote different positive constants;
- "$\to$" denoted strongly convergence, "$\rightharpoonup$" denoted weakly convergence.

2. Existence of minimizers for $I(M)$

This section is concerned with the proof of Theorem 1.2 on the existence of minimizers for (1.1). We first establish the subadditivity inequality of $I(M)$, and then prove Theorem 1.2 by applying the concentration-compactness principle. Now, we give the following Lemma about subadditivity.

Lemma 2.1. *Assume $m(x)$ satisfies* (A1), *then for any $M \in (0, \infty)$, it holds that*

$$I(M) < 0. \tag{2.1}$$

Moreover, we have the following strict subadditivity inequality

$$I(M) < I(\alpha) + I(M-\alpha) \quad \forall\, \alpha \in (0, M). \tag{2.2}$$

Proof. As for (2.1), set $u_\varsigma(x) := \varsigma^{N/2} u(\varsigma x)$, where $\varsigma > 0$ and $u \in H^s(\mathbb{R}^N)$ satisfies $\|u\|^2 = M$. One can deduce that, for any $M \in (0, \infty)$,

$$\begin{aligned} I(M) \le E(u_\varsigma) &= \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_\varsigma|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} m(x) |u_\varsigma|^{p+1} dx \\ &= 2\Big(\frac{\varsigma^{2s}}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx - \frac{\varsigma^{\frac{N(p-1)}{2}}}{p+1} \int_{\mathbb{R}^N} m(\frac{x}{\varsigma}) |u|^{p+1} dx \Big), \end{aligned}$$

it then follows that $E(u_\varsigma) < 0$ for $\varsigma > 0$ sufficiently small, because $\frac{N(p-1)}{2} < 2s$ and $m(x)$ satisfies (A1), thus, (2.1) holds. As for (2.2), for any $M \in (0, \infty)$ and $\theta \in \big(1, \frac{M}{\alpha}\big]$, we have

$$\begin{aligned} I(\theta\alpha) &= \inf_{u \in H^s(\mathbb{R}^N), \|u\|_2^2 = \theta\alpha} E(u) = \inf_{v \in H^s(\mathbb{R}^N), \|v\|_2^2 = \alpha} E(\theta^{1/2} v) \\ &= \inf_{v \in H^s(\mathbb{R}^N), \|v\|_2^2 = \alpha} \Big\{ \theta \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v|^2 dx - \frac{2\theta^{\frac{p+1}{2}}}{p+1} \int_{\mathbb{R}^N} m(\frac{x}{\varsigma}) |v|^{p+1} dx \Big\} \\ &= \inf_{v \in H^s(\mathbb{R}^N), \|v\|_2^2 = \alpha} \Big\{ \theta \Big[\int_{\mathbb{R}^N} |(-\Delta)^{s/2} v|^2 dx - \frac{2}{p+1} \int_{\mathbb{R}^N} m(\frac{x}{\varsigma}) |v|^{p+1} \Big] \\ &\quad + \frac{2(\theta - \theta^{\frac{p+1}{2}})}{p+1} \int_{\mathbb{R}^N} m(\frac{x}{\varsigma}) |v|^{p+1} dx \Big\} \\ &< \theta I(\alpha), \end{aligned}$$

where the last inequality holds because $\theta > 1$ and $p > 1$. This implies that for any $M \in (0, \infty)$,

$$I(\theta\alpha) < \theta I(\alpha) \quad \forall \alpha \in (0, M), \theta \in (1, \frac{M}{\alpha}]. \tag{2.3}$$

Furthermore, it follows from (2.3) that

$$I(M) = \frac{M-\alpha}{M} I\big(\frac{M}{M-\alpha}(M-\alpha) \big) + \frac{\alpha}{M} I\big(\frac{M}{\alpha} \cdot \alpha \big) < I(M-\alpha) + I(\alpha) \quad \forall \alpha \in (0, M).$$

Hence, the proof of Lemma 2.1 is complete. $\square$

Next, we introduce the concentration-compactness principle to fractional Sobolev spaces $H^s(\mathbb{R}^N)$, see for example [12].

Lemma 2.2. *Let $N \ge 2$, suppose $\{u_n\}_{n\ge1} \subset H^s(\mathbb{R}^N)$ and satisfies*

$$\int_{\mathbb{R}^N} |u_n|^2 dx = \rho > 0, \quad \sup_{n \ge 1} \|u_n\|_{H^s(\mathbb{R}^N)} < \infty.$$

Then there exists a subsequence $\{u_{n_k}\}_{k\ge1}$ for which one of the following properties holds.

(i) *Compactness: there exists a sequence $\{y_k\}_{k\ge1}$ in $\mathbb{R}^N$, such that, for any $\varepsilon > 0$, there exists $0 < r < \infty$ with*

$$\int_{|x-y_k|\le r} |u_{n_k}|^2 dx \ge \rho - \varepsilon.$$

(ii) *Vanishing: for all $r < \infty$, it follows that*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{|x-y|\le r} |u_{n_k}|^2 dx = 0.$$

(iii) *Dichotomy: there exist a constant $\beta \in (0, \rho)$ and two bounded sequences $\{v_k\}_{k\ge1}, \{w_k\}_{k\ge1} \subset H^s(\mathbb{R}^N)$ such that*

$$\begin{gathered} \operatorname{supp} v_k \cap \operatorname{supp} w_k = \emptyset, \\ |v_k| + |w_k| \le |u_{n_k}|, \\ \|v_k\|_2^2 \to \beta, \|w_k\|_2^2 \to (\rho - \beta) \quad \text{as } k \to \infty, \end{gathered}$$

$$\|u_{n_k} - v_k - w_k\|_q \to 0 \quad \textit{for } q \in [2, 2_s^*),$$
$$\liminf_{k\to\infty}\{\langle(-\Delta)^s u_{n_k}, u_{n_k}\rangle - \langle(-\Delta)^s v_k, v_k\rangle - \langle(-\Delta)^s w_k, w_k\rangle\} \geq 0.$$

Proof of the Theorem 1.2. We first claim that $-\infty < I(M) < 0$. For any given $M \in (0,\infty)$, assume $\|u\|_2^2 = M$. From Lemma 2.1, there holds that $I(M) < 0$. On the other hand, applying (A1) and Gagliardo-Nirenberg-Sobolev inequality (1.3) to $E(u)$, then yields that

$$\begin{aligned} E(u) &= \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx - \frac{2}{p+1}\int_{\mathbb{R}^N} m(x)|u|^{p+1}dx \\ &\geq \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx - \|Q\|_2^{1-p} M^{\frac{2s(p+1)-N(p-1)}{4s}} \Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx\Big)^{\frac{N(p-1)}{4s}} \\ &\geq -\frac{4s - N(p-1)}{4s}\Big(\frac{CN(p-1)}{4s}\Big)^{\frac{4s}{4s-N(p-1)}}, \end{aligned} \tag{2.4}$$

where $C = \|Q\|_2^{1-p} M^{\frac{2s(p+1)-N(p-1)}{4s}}$. This implies that $E(u)$ is bounded from below for any $M \in (0,\infty)$. Let $\{u_n\} \subset H^s(\mathbb{R}^N)$ be a minimizing sequence satisfies $\|u\|_2^2 = M$ and $\lim_{n\to\infty} E(u_n) = I(M)$, then (2.4) shows that $\{u_n\}$ is bounded uniformly in $H^s(\mathbb{R}^N)$. Moreover, since $-\infty < I(M) < 0$, we obtain $E(u_n) \leq \frac{I(M)}{2}$ for n sufficiently large. Therefore, we can deduce from (1.2) that

$$\int_{\mathbb{R}^N} m(x)|u_n|^{p+1}dx \geq \frac{(p+1)I(M)}{4}. \tag{2.5}$$

Applying Lemma 2.2 with $\rho = 1$, one can conclude that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ satisfies the compactness or the dichotomy or the vanishing.

We first prove that the vanishing does not occur. If not, according to vanishing Lemma, we know that $u_{n_k} \to 0$ in $L^{p+1}(\mathbb{R}^N)$. This is a contradiction with (2.5).

Next we prove that the dichotomy does not occur. If not, then Lemma 2.2(iii) shows that there exist two sequences $\{v_k\}, \{w_k\}$ such that $\liminf_{k\to\infty}(E(u_{n_k}) - E(v_k) - E(w_k)) \geq 0$, which implies

$$\limsup_{k\to\infty}(E(v_k) + E(w_k)) \leq I(M). \tag{2.6}$$

By direct calculation, it follows that

$$E(u) = \frac{1}{\xi^2}E(\xi u) + \frac{2(\xi^{p-1}-1)}{P+1}\int_{\mathbb{R}^N} m(x)|u|^{p+1}dx. \tag{2.7}$$

Set $\xi_k = \frac{1}{\|v_k\|_2}$, then $\|\xi_k v_k\|_2^2 = 1$, (2.7) shows that

$$E(v_k) \geq \frac{I(M)}{\xi_k^2} + \frac{2(\xi_k^{p-1}-1)}{P+1}\int_{\mathbb{R}^N} m(x)|v_k|^{p+1}dx. \tag{2.8}$$

Similarly, we have

$$E(w_k) \geq \frac{I(M)}{\gamma_k^2} + \frac{2(\gamma_k^{p-1}-1)}{P+1}\int_{\mathbb{R}^N} m(x)|w_k|^{p+1}dx, \tag{2.9}$$

where $\gamma_k = 1/\|w_k\|_2$. By the fact that $\xi_k \to \beta^{-1/2}$ and $\gamma_k \to (1-\beta)^{-1/2}$ as $k \to \infty$, we deduce from (2.5),(2.8),(2.9) and Lemma 2.2(iii) that

$$\begin{aligned} \liminf_{k\to\infty}(E(v_k) + E(w_k)) &\geq I(M) + \frac{2(\zeta-1)}{p+1}\liminf_{k\to\infty}\int_{\mathbb{R}^N} m(x)|u_{n_k}|^{p+1}dx \\ &\geq I(M) - \frac{(\zeta-1)}{2}I(M) > I(M), \end{aligned} \tag{2.10}$$

where $\zeta := \min\{\beta^{-1/2}, (1-\beta)^{-1/2}\} > 1$, due to $\beta \in (0,1)$ and $I(M) < 0$. Combining (2.6) and (2.10), we can conclude that the dichotomy does not occur.

Now Lemma 2.2(i) shows that there exists a subsequence of $\{u_{n_k}\}$ (still denoted by $\{u_{n_k}\}$) and some $\{y_k\} \subset (\mathbb{R}^N)$, such that $\hat{u}_{n_k} := u_{n_k}(\cdot + y_k)$ satisfies

$$\begin{gathered} \hat{u}_{n_k} \rightharpoonup u_0 \quad \text{in } H^s(\mathbb{R}^N) \quad \text{for some } u_0 \in H^s(\mathbb{R}^N), \\ \hat{u}_{n_k} \to u_0 \quad \text{in } L^q(\mathbb{R}^N) \quad \text{for } q \in [2, 2_s^*). \end{gathered} \tag{2.11}$$

We can see that
$$\lim_{k\to\infty}\int_{\mathbb{R}^N} m(x)|\hat{u}_{n_k}|^{p+1}dx=\int_{\mathbb{R}^N} m(x)|u_0|^{p+1}dx.$$
By the weakly lower semicontinuity, we deduce that
$$E(u_0)\le \lim_{k\to\infty}E(\hat{u}_{n_k}), \tag{2.12}$$
we observe that $I(M)\le \lim_{k\to\infty}E(\hat{u}_{n_k})$ from the definition of $I(M)$. If $I(M)=\lim_{k\to\infty}E(\hat{u}_{n_k})$, using (2.12) we have
$$I(M)\le E(u_0)\le \lim_{k\to\infty}E(\hat{u}_{n_k})=I(M), \tag{2.13}$$
it follows that u_0 is a minimizer of $I(M)$ for any $M\in(0,\infty)$. On the other hand, $I(M)<\lim_{k\to\infty}E(\hat{u}_{n_k})$, we claim that $\{y_k\}$ is bounded in $\mathbb{R}^N$. Otherwise, assume $y_k\to\infty$, then we have
$$\begin{aligned}
I(M)&=\lim_{k\to\infty}E(u_{n_k})=\lim_{k\to\infty}\Big\{\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_{n_k}|^2dx-\frac{2}{p+1}\int_{\mathbb{R}^N}m(x)|u_{n_k}|^{p+1}dx\Big\}\\
&=\lim_{k\to\infty}\Big\{\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{u}_{n_k}|^2dx-\frac{2}{p+1}\int_{\mathbb{R}^N}m(x+y_k)|\hat{u}_{n_k}|^{p+1}dx\Big\}\\
&\ge \lim_{k\to\infty}E(\hat{u}_{n_k})+\frac{2}{p+1}\int_{\mathbb{R}^N}(m(x)-m(x+y_k))|\hat{u}_{n_k}|^{p+1}dx\\
&\ge \lim_{k\to\infty}E(\hat{u}_{n_k}),
\end{aligned}$$
where the last inequality holds because of $\inf_{x\in\mathbb{R}^N}m(x)=\lim_{|x|\to\infty}m(x)$. This is a contradiction, thus the claim holds. Passing to a subsequence if necessary, we have $\lim_{k\to\infty}y_k\to y_0$ for some $y_0\in\mathbb{R}^N$. It follows from (2.11) that $u_{n_k}\to u_0$ in $L^q(\mathbb{R}^N)$ with $q\in[2,2_s^*)$. This yields that
$$\lim_{k\to\infty}\int_{\mathbb{R}^N} m(x)|u_{n_k}|^{p+1}dx=\int_{\mathbb{R}^N} m(x)|u_0(x-y_0)|^{p+1}dx.$$
Similar to (2.12) and (2.13), we can deduce that
$$I(M)\le E(u_0(\cdot-y_0))\le \lim_{k\to\infty}E(u_{n_k}),$$
which implies that $u_0(\cdot-y_0)$ is a minimizer of $I(M)$. Then the proof is complete. □

3. Mass concentration

In this section, we prove Theorem 1.3 on the concentration behavior of minimizers for $I(M_k)$ with $M_k\to\infty$ as $k\to\infty$. We first establish the following Theorem.

Theorem 3.1. *Suppose $m(x)$ satisfies* (A1), (A2). *Let u_k be a nonnegative minimizer of I_{M_k} with $M_k\to\infty$ as $k\to\infty$. Then passing to a subsequence if necessary, u_k has a unique maximum point $\bar{z}_k$ as k is large enough, and $\bar{z}_k$ satisfies* $\lim_{k\to\infty}\bar{z}_k=0$. *Moreover, there also holds that*
$$\lim_{k\to\infty}\hat{\varepsilon}_k^{N/2}u_k(\hat{\varepsilon}_kx+\bar{z}_k)=(a^*)^{-\frac{2s}{4s-N(p-1)}}Q\Big((a^*)^{-\frac{p-1}{4s-N(p-1)}}x\Big)\quad in\ H^s(\mathbb{R}^N), \tag{3.1}$$
where $\hat{\varepsilon}_k:=M_k^{-\frac{p-1}{4s-N(p-1)}}$, $a^*=\|Q\|_2^2$ and $Q(x)$ is the unique radially symmetric positive solution of (1.5).

3.1. **Energy estimates of I_M.** This section is aimed to establishing the refined energy estimates of I_M by employing the analysis of $\tilde{I}_M$ defined in (1.11). As for the estimate of $\tilde{I}_M$, we have the following Lemma.

Lemma 3.2. *Suppose $\tilde{u}_M$ is a nonnegative minimizer of $\tilde{I}_M$. Then*
$$\tilde{I}_M=-\lambda\Big(\frac{M}{a_*}\Big)^{\frac{2s(p-1)}{4s-N(p-1)}}, \tag{3.2}$$
and up to translations, $\tilde{u}_M$ satisfies
$$\tilde{u}_M=\frac{1}{\sqrt{a^*}}\tilde{\alpha}_M^{N/2}Q(\alpha_Mx), \tag{3.3}$$

where $a^* = \|Q\|_2^2$, $Q(x)$ *is the unique radially symmetric positive solution of* (1.5),

$$\tilde{\alpha}_M := (\frac{M}{a^*})^{\frac{p-1}{4s-N(p-1)}}(\frac{N(p-1)}{4s})^{\frac{2}{4s-N(p-1)}}$$

and λ *is defined by*

$$\lambda := \frac{4s-N(p-1)}{4s}\Big(\frac{N(p-1)}{4s}\Big)^{\frac{N(p-1)}{4s-N(p-1)}}. \tag{3.4}$$

Proof. Suppose $\tilde{u}_M$ is a nonnegative minimizer of $\tilde{I}_M$, and suppose $\tilde{u}_1$ is a nonnegative minimizer of $\tilde{I}_1$. First we claim that

$$\tilde{I}_M = \alpha_M^{2s}\tilde{I}_1, \quad \tilde{u}_M = \alpha_M^{N/2}\tilde{u}_1(\alpha_M x) \quad \text{with} \quad \alpha_M = M^{\frac{p-1}{4s-N(p-1)}}. \tag{3.5}$$

Indeed, set $\tilde{v}_1 := \alpha_M^{-\frac{N}{2}}\tilde{u}_M(\alpha_M^{-1}x)$. Simple calculations show that $\|\tilde{v}_1\|_2^2 = 1$ and

$$\begin{aligned}
\tilde{I}_M &= \tilde{E}_M(\tilde{u}_M)\\
&= \int_{\mathbb{R}^N}|(-\Delta)^{s/2}\tilde{u}_M|^2dx - \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}|\tilde{u}_M|^{p+1}dx\\
&= \alpha_M^{2s}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}v_1|^2dx - \frac{2M^{\frac{p-1}{2}}}{p+1}\alpha_M^{\frac{N(p-1)}{2}}\int_{\mathbb{R}^N}|\tilde{v}_1|^{p+1}dx\\
&= M^{\frac{2s(p-1)}{4s-N(p-1)}}\Big(\int_{\mathbb{R}^N}|(-\Delta)^{s/2}v_1|^2dx - \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}|\tilde{v}_1|^{p+1}dx\Big)\\
&\geq M^{\frac{2s(p-1)}{4s-N(p-1)}}\tilde{I}_1
\end{aligned} \tag{3.6}$$

Similarly, setting $\tilde{v}_M := \alpha_M^{N/2}\tilde{u}_1(\alpha_M x)$, we have

$$\tilde{I}_M \leq \tilde{E}_M(\tilde{u}_M) = \alpha_M^{2s}\tilde{I}_1. \tag{3.7}$$

From (3.6) and (3.7), we can deduce that (3.5) holds. Next we prove that

$$\tilde{I}_1 = -\lambda\Big(\frac{1}{a^*}\Big)^{\frac{2s(p-1)}{4s-N(p-1)}} \tag{3.8}$$

and $\tilde{u}_1$ satisfies

$$\tilde{u}_1 = \Big(\frac{N(p-1)}{4s}\Big)^{\frac{N}{4s-N(p-1)}}\sqrt{a^*}^{-\frac{4s}{4s-N(p-1)}}Q\Big(\Big(\frac{N(p-1)}{4s}\Big)^{\frac{2}{4s-N(p-1)}}\sqrt{a^*}^{-\frac{2(p-1)}{4s-N(p-1)}}x\Big). \tag{3.9}$$

Take a test function $\tilde{v}_\varepsilon = \varepsilon^{N/2}\tilde{v}_0(\varepsilon x)$, where $0 < \tilde{v}_0 \in H^s(\mathbb{R}^N)$ satisfies $\|\tilde{v}_0\|_2^2 = 1$ and $\varepsilon > 0$ is a positive constant, we have

$$\tilde{I}_1 \leq \tilde{E}_1(\tilde{v}_\varepsilon) = \varepsilon^{2s}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\tilde{v}_0|^2dx - \frac{2}{p+1}\varepsilon^{\frac{N(p-1)}{2}}\int_{\mathbb{R}^N}|\tilde{v}_0|^{p+1}dx < 0,$$

when ε is small enough, due to $2s > \frac{N(p-1)}{2}$. Note that the minimizer $\tilde{u}_1$ of $\tilde{I}_1$ satisfies

$$(-\Delta)^s\tilde{u}_1 = \tilde{\mu}_1\tilde{u}_1 + \tilde{u}_1^p \quad \text{in } \mathbb{R}^N, \tag{3.10}$$

where $\tilde{\mu}_1$ is a suitable Lagrange multiplier. It follows from (1.12) and (3.10) that

$$\tilde{\mu}_1 = -\tilde{I}_1 + \frac{p-1}{p+1}\int_{\mathbb{R}^N}|\tilde{u}_1|^{p+1}dx > 0. \tag{3.11}$$

Similar to the proof of [35, Proposition 4.1], we know $\tilde{u}_1 > 0$. In view of (1.6) and (3.10), it follows that

$$\tilde{u}_1(x) = \tilde{\mu}_1^{\frac{1}{p-1}}\Big(\frac{4s}{2s(p+1)-N(p-1)}\Big)^{\frac{1}{p-1}}Q\Big(\tilde{\mu}_1^{\frac{1}{2s}}\Big(\frac{N(p-1)}{2s(p+1)-N(p-1)}\Big)^{\frac{1}{2s}}x\Big). \tag{3.12}$$

Furthermore, since $\|\tilde{u}_1\|_2^2 = 1$, one can deduce that $\tilde{\mu}_1$ satisfies

$$\tilde{\mu}_1 = \Big(\frac{N(p-1)}{4s}\Big)^{\frac{4s}{4s-N(p-1)}}\frac{2s(p+1)-N(p-1)}{N(p-1)}\sqrt{a^*}^{-\frac{4s(p-1)}{4s-N(p-1)}},$$

from which and (3.12), we can deduce that (3.9) holds. Then (1.7), (1.12) and (3.9) show that (3.8) holds. Thus, combining (3.5) and (3.8), we can see that (3.2), (3.3) hold. □

Based on Lemma 3.2, we can obtain the following energy estimates of I_M.

Lemma 3.3. *Suppose $m(x)$ satisfies (A1). Then we have*

$$\lim_{n\to\infty} I_M = -\lambda\Big(\frac{M}{a_*}\Big)^{\frac{2s(p-1)}{4s-N(p-1)}}, \tag{3.13}$$

where $a^* = \|Q\|_2^2$ and $Q(x)$ is the unique radially symmetric positive solution of (1.5)*, λ is given by* (3.4).

Proof. Let u_M be a nonnegative minimizer of I_M. According to the definition of I_M and $\tilde{I}_M$, we deduce from Lemma 3.2 that

$$\begin{aligned} I_M &= \tilde{E}_M(u_M) + \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}(1-m(x))|u_M|^{p+1}dx \\ &\geq \tilde{I}_M = -\lambda\Big(\frac{M}{a_*}\Big)^{\frac{2s(p-1)}{4s-N(p-1)}} \quad \text{as } M\to\infty. \end{aligned} \tag{3.14}$$

On the other hand, set a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N)$, such that $\varphi(x) = 1$ for $|x| \leq 1$, $\varphi(x) = 0$ for $|x| \geq 2$, $0 \leq \varphi(x) \leq 1$ and $|\nabla\varphi| \leq 2$. Define

$$u_\tau(x) := \frac{A_\tau}{\|Q\|_2^2}\tau^{N/2}\varphi(x)Q(\tau x), \tag{3.15}$$

where $\tau > 0$, A_τ is chosen so that $\|u_\tau\|_2^2 = 1$. Applying (1.4), we have

$$1 \leq A_\tau^2 \leq 1 + C\tau^{-N-4s} \quad \text{as } \tau\to\infty. \tag{3.16}$$

In fact, since $\|u_\tau\|_2^2 = 1$, it follows that

$$1 = \frac{A_\tau^2}{\|Q\|_2^2}\int_{\mathbb{R}^N}\varphi^2(\frac{x}{\tau})Q^2(x)dx \leq \frac{A_\tau^2}{\|Q\|_2^2}\|Q\|_2^2 = A_\tau^2,$$

which implies the left inequality of (3.16). On the other hand,

$$1 = \frac{A_\tau^2}{\|Q\|_2^2}\int_{\mathbb{R}^N}\varphi^2(\frac{x}{\tau})Q^2(x)dx \geq \frac{A_\tau^2}{\|Q\|_2^2}\int_{B_\tau(0)}Q^2(x)dx,$$

thus, we have

$$\begin{aligned} A_\tau^2 &\leq \frac{\|Q\|_2^2}{\int_{B_\tau(0)}Q^2(x)dx} \\ &= \frac{\int_{B_\tau(0)}Q^2(x)dx + \int_{\mathbb{R}^N\setminus B_\tau(0)}Q^2(x)dx}{\int_{B_\tau(0)}Q^2(x)dx} \\ &\leq 1 + \frac{\int_{\mathbb{R}^N\setminus B_\tau(0)}|\frac{C_2}{1+|x|^{N+2s}}|^2(x)dx}{\int_{B_\tau(0)}Q^2(x)dx} \\ &\leq 1 + C\tau^{-N-4s}. \end{aligned}$$

Furthermore, $A_\tau^2 \to 1$ as $\tau\to\infty$. Combining (1.7) and (3.16), we have

$$\begin{aligned} I_M &\leq E_M(u_\tau) \\ &= \int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_\tau|^2dx - \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}|m(x)u_\tau|^{p+1}dx \\ &= \int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_\tau|^2dx - \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}|u_\tau|^{p+1}dx + \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}(1-m(x))|u_\tau|^{p+1}dx \\ &= \frac{A_\tau^2\tau^{2s}}{\|Q\|_2^2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}Q(x)|^2dx - \frac{2M^{\frac{p-1}{2}}}{p+1}\frac{A_\tau^{p+1}\tau^{\frac{N(p-1)}{2}}}{\|Q\|_2^{p+1}}\int_{\mathbb{R}^N}Q(x)^{p+1}dx \end{aligned}$$

$$
\begin{aligned}
&+\frac{2M^{\frac{p-1}{2}}\tau^{-r-2}}{p+1}\frac{A_\tau^{p+1}\tau^{\frac{N(p-1)}{2}}}{\|Q\|_2^{p+1}}\int_{\mathbb{R}^N}Q(x)^{p+1}dx\\
&\le (1+C\tau^{-N-4s})\tau^{2s}-(1-\tau^{-r-2})\Big(\frac{M}{a^*}\Big)^{\frac{p-1}{2}}\tau^{\frac{N(p-1)}{2}}\quad \text{as } \tau\to\infty,
\end{aligned}
$$

where $r>0$ is defined in (A2). Take $\tau=\big(\frac{N(p-1)}{4s}\big)^{\frac{2}{4s-N(p-1)}}\left(\frac{M}{a^*}\right)^{\frac{p-1}{4s-N(p-1)}}$, then yields that

$$
\begin{aligned}
I_M &\le \Big(\frac{N(p-1)}{4s}\Big)^{\frac{4s}{4s-N(p-1)}}\Big(\frac{M}{a^*}\Big)^{\frac{2s(p-1)}{4s-N(p-1)}}-\Big(\frac{N(p-1)}{4s}\Big)^{\frac{N(p-1)}{4s-N(p-1)}}\Big(\frac{M}{a^*}\Big)^{\frac{2s(p-1)}{4s-N(p-1)}}+o(1)\\
&=-\lambda\Big(\frac{M}{a^*}\Big)^{\frac{2s(p-1)}{4s-N(p-1)}}\quad \text{as } M\to\infty,
\end{aligned}
\tag{3.17}
$$

where λ is given by (3.4). Therefore, (3.13) follows from (3.14) and (3.17) directly, and Lemma 3.3 is then proved. $\square$

3.2. **Blow-up analysis.** The main purpose of this subsection is to establish blow-up analysis of I_M with $M\to\infty$. Motivated by Guo [18, Lemma 2.2] and Maeda [27, Lemma 4.2], combining a new trial function and the polynomial decay of Q, we first obtain the following Lemma.

Lemma 3.4. *Let u_M be a nonnegative minimizer of I_M, then we have*

$$0\le I_M-\tilde I_M\to 0\quad \text{as } M\to\infty.$$

Proof. Set $\hat u_M=\frac{A_M}{\sqrt{a^*}}\tilde\alpha_M^{N/2}\varphi(\tilde\alpha_M^{-t})Q(\tilde\alpha_M x)$, $t>0$, where $\tilde\alpha_M$ and φ are given in Lemma 3.2, $a^*=\|Q\|_2^2$, $Q(x)$ is the unique radially symmetric positive solution of (1.5) and $A_M>0$ is chosen so that $\|\hat u_M\|_2^2=1$. As in Lemma 3.2, we have

$$1\le A_M^2\le 1+\tilde C\tilde\alpha_M^{-(t+1)(N+4s)}. \tag{3.18}$$

Using the nonlocal Leibniz rule, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat u_M|^2dx-\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\tilde u_M|^2dx\\
&=\frac{A_M^2\tilde\alpha_M^{2s}}{a^*}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\varphi(\tilde\alpha_M^{-t-1}x)Q(\tilde\alpha_M x)|^2dx-\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\tilde u_M|^2dx\\
&=\frac{A_M^2\tilde\alpha_M^{2s}}{a^*}\int_{\mathbb{R}^N}|Q(x)(-\Delta)^{s/2}\varphi(\tilde\alpha_M^{-t-1}x)+\varphi(\tilde\alpha_M^{-t-1}x)(-\Delta)^{s/2}Q(x)\\
&\quad -B(\varphi(\tilde\alpha_M^{-t-1}x),Q(x))|^2dx-\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\tilde u_M|^2dx\\
&:=T_1+T_2+T_3+T_4+T_5+T_6,
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{gathered}
T_1:=\frac{A_M^2\tilde\alpha_M^{2s}}{a^*}\int_{\mathbb{R}^N}\varphi^2(\tilde\alpha_M^{-t-1}x)|(-\Delta)^{s/2}Q(x)|^2dx-\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\tilde u_M|^2dx,\\
T_2:=\frac{A_M^2\tilde\alpha_M^{2s}}{a^*}\int_{\mathbb{R}^N}Q^2(x)|(-\Delta)^{s/2}\varphi(\tilde\alpha_M^{-t-1}x)|^2dx,\\
T_3:=\frac{A_M^2\tilde\alpha_M^{2s}}{a^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}B^2(\varphi(\tilde\alpha_M^{-t-1}x),Q(x)),\\
T_4:=2\frac{A_M^2\tilde\alpha_M^{2s}}{a^*}\int_{\mathbb{R}^N}Q(x)\varphi(\tilde\alpha_M^{-t-1}x)(-\Delta)^{s/2}Q(x)(-\Delta)^{s/2}\varphi(\tilde\alpha_M^{-t-1}x)dx,\\
T_5:=-2\frac{A_M^2\tilde\alpha_M^{2s}}{a^*}\int_{\mathbb{R}^N}Q(x)(-\Delta)^{s/2}\varphi(\tilde\alpha_M^{-t-1}x)B(\varphi(\tilde\alpha_M^{-t-1}x),Q(x))dx,\\
T_6:=-2\frac{A_M^2\tilde\alpha_M^{2s}}{a^*}\int_{\mathbb{R}^N}\varphi(\tilde\alpha_M^{-t-1}x)(-\Delta)^{s/2}Q(x)B(\varphi(\tilde\alpha_M^{-t-1}x),Q(x))dx,
\end{gathered}
$$

and

$$B(\varphi(\tilde{\alpha}_M^{-t-1}x), Q(x)) = C_s P.V. \int_{\mathbb{R}^N} \frac{(\varphi(\tilde{\alpha}_M^{-t-1}x) - \varphi(\tilde{\alpha}_M^{-t-1}y))(Q(x) - Q(y))}{|x-y|^{N+s}} dy.$$

To estimate $T_1 - T_6$, two inequalities are established below. By direct calculation, we have

$$\begin{aligned}
&|(-\Delta)^{s/2}\varphi(\tilde{\alpha}_M^{-t-1}x)| \\
&= C_s P.V. \Big|\int_{\mathbb{R}^N} \frac{\varphi(\tilde{\alpha}_M^{-t-1}x) - \varphi(\tilde{\alpha}_M^{-t-1}y)}{|x-y|^{N+s}} dy\Big| \\
&\le C\tilde{\alpha}_M^{-t-1} \int_{|x-y|\le \tilde{\alpha}_M^{t+1}} \frac{|\nabla\varphi(\tilde{\alpha}_M^{-t-1}\xi)|}{|x-y|^{N+s}} dy + C\int_{|x-y|\ge \tilde{\alpha}_M^{t+1}} \frac{1}{|x-y|^{N+s}} dy \\
&\le C\tilde{\alpha}_M^{-s(t+1)},
\end{aligned} \tag{3.20}$$

where $\xi = y + \theta(x-y)$ with $\theta \in (0,1)$. By the Hölder inequality and (3.20), we obtain

$$\begin{aligned}
&B(\varphi(\tilde{\alpha}_M^{-t-1}x), Q(x)) \\
&= C_s P.V. \int_{\mathbb{R}^N} \frac{(\varphi(\tilde{\alpha}_M^{-t-1}x) - \varphi(\tilde{\alpha}_M^{-t-1}y))(Q(x) - Q(y))}{|x-y|^{N+s}} dy \\
&\le C\Big(\int_{\mathbb{R}^N} \frac{|\varphi(\tilde{\alpha}_M^{-t-1}x) - \varphi(\tilde{\alpha}_M^{-t-1}y)|^2}{|x-y|^{N+s}} dy\Big)^{1/2} \Big(\int_{\mathbb{R}^N} \frac{|Q(x) - Q(y)|^2}{|x-y|^{N+s}} dy\Big)^{1/2} \\
&\le 2C\Big(\int_{\mathbb{R}^N} \frac{|\varphi(\tilde{\alpha}_M^{-t-1}x) - \varphi(\tilde{\alpha}_M^{-t-1}y)|}{|x-y|^{N+s}} dy\Big)^{1/2} \Big(\int_{\mathbb{R}^N} \frac{|Q(x) - Q(y)|^2}{|x-y|^{N+s}} dy\Big)^{1/2} \\
&\le C\tilde{\alpha}_M^{-\frac{s}{2}(t+1)} \Big(\int_{\mathbb{R}^N} \frac{|Q(x) - Q(y)|^2}{|x-y|^{N+s}} dy\Big)^{1/2}
\end{aligned} \tag{3.21}$$

In view of (1.7), (3.3) and (3.18), it follows that

$$\begin{aligned}
T_1 &= \frac{(A_M^2 - 1)\tilde{\alpha}_M^{2s}}{a^*} \int_{\mathbb{R}^N} \varphi^2(\tilde{\alpha}_M^{-t-1}x)|(-\Delta)^{s/2}Q(x)|^2 dx \\
&\quad + \frac{\tilde{\alpha}_M^{2s}}{a^*} \int_{\mathbb{R}^N} (\varphi^2(\tilde{\alpha}_M^{-t-1}x) - 1)|(-\Delta)^{s/2}Q(x)|^2 dx \\
&\quad + \frac{\tilde{\alpha}_M^{2s}}{a^*} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}Q(x)|^2 dx - \int_{\mathbb{R}^N} |(-\Delta)^{s/2}\tilde{u}_M|^2 dx \\
&\le \frac{(A_M^2 - 1)\tilde{\alpha}_M^{2s}}{a^*} \int_{\mathbb{R}^N} \varphi^2(\tilde{\alpha}_M^{-t-1}x)|(-\Delta)^{s/2}Q(x)|^2 dx \\
&\le C\tilde{\alpha}_M^{2s-(t+1)(N+4s)} \quad \text{as } M \to \infty.
\end{aligned} \tag{3.22}$$

Using (1.7), (3.18) and (3.20), we have

$$|T_2| \le C\frac{A_M^2\tilde{\alpha}_M^{2s}}{a^*}\tilde{\alpha}_M^{-2s(t+1)} \int_{\mathbb{R}^N} |Q|^2 dx \le C(1 + \tilde{C}\tilde{\alpha}_M^{-(t+1)(N+4s)})\tilde{\alpha}_M^{-2st} \quad \text{as } M \to \infty. \tag{3.23}$$

Applying (1.7), (3.18),(3.21) and [29, Proposition 3.4], we obtain

$$\begin{aligned}
|T_3| &\le C\frac{A_M^2\tilde{\alpha}_M^{2s}}{a^*}\tilde{\alpha}_M^{-s(t+1)} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|Q(x) - Q(y)|^2}{|x-y|^{N+s}}\, dy\, dx \\
&\le C\frac{A_M^2\tilde{\alpha}_M^{s(1-t)}}{a^*} \int_{\mathbb{R}^N} |\xi|^s|\check{Q}(\xi)|^2 d\xi \\
&\le C\frac{1 + \tilde{C}\tilde{\alpha}_M^{-(t+1)(N+4s)}}{a^*}\tilde{\alpha}_M^{s(1-t)} \int_{\mathbb{R}^N} (1 + |\xi|^{2s})|\check{Q}(\xi)|^2 d\xi \\
&\le C(1 + \tilde{C}\tilde{\alpha}_M^{-(t+1)(N+4s)})\tilde{\alpha}_M^{s(1-t)}
\end{aligned} \tag{3.24}$$

as $M \to \infty$, where $\check{Q}$ denote the Fourier transform of Q. By the Hölder inequality, (1.7), (3.18) and (3.20), we obtain that

$$\begin{aligned} |T_4| &\leq C\frac{A_M^2\tilde{\alpha}_M^{2s}}{a^*}\Big(\int_{\mathbb{R}^N} Q^2(x)|(-\Delta)^{s/2}\varphi(\tilde{\alpha}_M^{-t-1}x)|^2dx\Big)^{1/2} \\ &\quad\times\Big(\int_{\mathbb{R}^N}\varphi^2(\tilde{\alpha}_M^{-t-1}x)|(-\Delta)^{s/2}Q|^2dx\Big)^{1/2} \\ &\leq C\frac{A_M^2\tilde{\alpha}_M^{2s}}{a^*}\Big(C_3\tilde{\alpha}_M^{-2s(t+1)}\int_{\mathbb{R}^N} Q^2(x)dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N}|(-\Delta)^{s/2}Q|^2dx\Big)^{1/2} \\ &\leq C(1+\tilde{C}\tilde{\alpha}_M^{-(t+1)(N+4s)})\tilde{\alpha}_M^{s(1-t)} \quad \text{as } M\to\infty. \end{aligned} \tag{3.25}$$

From the Hölder inequality, (1.7), (3.18), (3.20) and (3.21), it follows that

$$\begin{aligned} |T_5| &\leq C\frac{A_M^2\tilde{\alpha}_M^{2s}}{a^*}\Big(\int_{\mathbb{R}^N} Q^2(x)|(-\Delta)^{s/2}\varphi(\tilde{\alpha}_M^{-t-1}x)|^2dx\Big)^{1/2} \\ &\quad\times\Big(\int_{\mathbb{R}^N} B^2(\varphi(\tilde{\alpha}_M^{-t-1}x),Q(x))dx\Big)^{1/2} \\ &\leq C(1+\tilde{C}\tilde{\alpha}_M^{-(t+1)(N+4s)})\tilde{\alpha}_M^{2s}\tilde{\alpha}_M^{-s(t+1)}\tilde{\alpha}_M^{-\frac{s(t+1)}{2}} \\ &\leq C(1+\tilde{C}\tilde{\alpha}_M^{-(t+1)(N+4s)})\tilde{\alpha}_M^{s(\frac{1}{2}-\frac{3}{2}t)} \quad \text{as } M\to\infty. \end{aligned} \tag{3.26}$$

Similarly,

$$\begin{aligned} |T_6| &\leq C\frac{A_M^2\tilde{\alpha}_M^{2s}}{a^*}\Big(\int_{\mathbb{R}^N}\varphi^2(\tilde{\alpha}_M^{-t-1}x)|(-\Delta)^{s/2}Q|^2dx\Big)^{1/2}\Big(\int_{\mathbb{R}^N} B^2(\varphi(\tilde{\alpha}_M^{-t-1}x),Q(x))dx\Big)^{1/2} \\ &\leq C(1+\tilde{C}\tilde{\alpha}_M^{-(t+1)(N+4s)})\tilde{\alpha}_M^{2s}\tilde{\alpha}_M^{-\frac{s(t+1)}{2}} \\ &\leq C(1+\tilde{C}\tilde{\alpha}_M^{-(t+1)(N+4s)})\tilde{\alpha}_M^{s(\frac{3}{2}-\frac{1}{2}t)} \quad \text{as } M\to\infty. \end{aligned} \tag{3.27}$$

By the fact that $t>3$, we then deduce from (3.19), (3.22)-(3.27) that

$$\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{u}_M|^2dx-\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\tilde{u}_M|^2dx\to 0 \quad \text{as } M\to\infty. \tag{3.28}$$

By (1.4) and (3.18), we obtain that

$$\begin{aligned} &M^{\frac{p-1}{2}}\frac{A_M^{p+1}\tilde{\alpha}_M^{\frac{N(p-1)}{2}}}{\sqrt{a^*}^{p+1}}\int_{\mathbb{R}^N} m(x)(1-\varphi^{p+1}(\tilde{\alpha}_M^{-t-1}x))Q^{p+1}(x)dx \\ &\leq M^{\frac{p-1}{2}}\frac{A_M^{p+1}\tilde{\alpha}_M^{\frac{N(p-1)}{2}}}{\sqrt{a^*}^{p+1}}\int_{|\tilde{\alpha}_M^{-t-1}x\geq 1|} Q^{p+1}(x)dx \\ &\leq C(1+\tilde{C}\tilde{\alpha}_M^{-(t+1)(N+4s)})^{\frac{p+1}{2}}(a^*)^{-1}\Big(\frac{M}{a^*}\Big)^{\frac{p-1}{2}}A_M^{p+1}\tilde{\alpha}_M^{\frac{N(p-1)}{2}}\tilde{\alpha}_M^{(t+1)[N-(N+2s)(p+1)]} \\ &\leq C\tilde{\alpha}_M^{2s+(t+1)[N-(N+2s)(p+1)]}, \end{aligned}$$

then from (3.3) and (M_2), we conclude that

$$\begin{aligned} &M^{\frac{p-1}{2}}\int_{\mathbb{R}^N} m(x)|\hat{u}_M|^{p+1}dx \\ &=\Big(\frac{M}{a^*}\Big)^{\frac{p-1}{2}}\frac{A_M^{p+1}\tilde{\alpha}_M^{\frac{N(p-1)}{2}}}{a^*}\int_{\mathbb{R}^N} m(\tilde{\alpha}_M^{-1}x)\varphi^{p+1}(\tilde{\alpha}_M^{-t-1}x)Q^{p+1}(x)dx \\ &=\frac{4s}{N(p-1)}\frac{A_M^{p+1}\tilde{\alpha}_M^{2s}}{a^*}\int_{\mathbb{R}^N}(m(\tilde{\alpha}_M^{-1}x)-1)\varphi^{p+1}(\tilde{\alpha}_M^{-t-1}x)Q^{p+1}(x)dx \\ &\quad+\frac{4s}{N(p-1)}\frac{A_M^{p+1}\tilde{\alpha}_M^{2s}}{a^*}\int_{\mathbb{R}^N}(\varphi^{p+1}(\tilde{\alpha}_M^{-t-1}x)-1)Q^{p+1}(x)\,dx \end{aligned}$$

$$
\begin{aligned}
&+ \frac{4s}{N(p-1)}\frac{A_M^{p+1}\tilde{\alpha}_M^{2s}}{a^*}\int_{\mathbb{R}^N} Q^{p+1}(x)dx \\
&\geq -C_4 A_M^{p+1}\tilde{\alpha}_M^{2s}\tilde{\alpha}_M^{-(r+2)} - C_5 A_M^{p+1}\tilde{\alpha}_M^{2s}\tilde{\alpha}_M^{(t+1)[N-(N+2s)(p+1)]} \\
&+ \frac{4s}{N(p-1)}\frac{A_M^{p+1}\tilde{\alpha}_M^{2s}}{a^*}\Big(\frac{M}{a^*}\Big)^{\frac{p-1}{2}}\frac{N(p-1)}{4s}a^{*\,\frac{p-1}{2}}\int_{\mathbb{R}^N}|\tilde{u}_M|^{p+1}dx \\
&= -C_4\tilde{\alpha}_M^{2s-(r+2)} - C_5\tilde{\alpha}_M^{2s+(t+1)[N-(N+2s)(p+1)]} + M^{\frac{p-1}{2}}\int_{\mathbb{R}^N}|\tilde{u}_M|^{p+1}dx \quad \text{as } M\to\infty.
\end{aligned}
$$

which implies that

$$
M^{\frac{p-1}{2}}\int_{\mathbb{R}^N}|\tilde{u}_M|^{p+1}dx - M^{\frac{p-1}{2}}\int_{\mathbb{R}^N}m(x)|\hat{u}_M|^{p+1}dx \to 0 \quad \text{as } M\to\infty. \tag{3.29}
$$

Moreover

$$
\begin{aligned}
0 &\leq I_M - \tilde{I}_M \\
&\leq E_M(\hat{u}_M) - \tilde{E}_M(\tilde{u}_M) \\
&= \int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{u}_M|^2dx - \int_{\mathbb{R}^N}|(-\Delta)^{s/2}\tilde{u}_M|^2dx \\
&\quad + \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}|\tilde{u}_M|^{p+1}dx - \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}m(x)|\hat{u}_M|^{p+1}dx, \quad \text{for large } k.
\end{aligned} \tag{3.30}
$$

it follows from (3.28)-(3.30) that $0 \leq I_M - \tilde{I}_M \to 0$ as $M\to\infty$. Thus, we completes the proof. □

Remark 3.5. Obviously, Lemma 3.4 shows that

$$
I_M(u_M) - \tilde{I}_M(u_M) = \frac{2M^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}(1-m(x))|u_M|^{p+1}dx \to 0.
$$

Let u_k be a nonnegative minimizer of I_{M_k} with $M_k\to\infty$ as $k\to\infty$. we define

$$
\hat{\varepsilon}_k := M_k^{-\frac{p-1}{4s-N(p-1)}} \quad \text{with } M_k\to\infty \text{ as } k\to\infty; \tag{3.31}
$$

$$
\hat{w}_k := \hat{\varepsilon}_k^{N/2}u_k(\hat{\varepsilon}_k x) \quad \text{with } M_k\to\infty \text{ as } k\to\infty. \tag{3.32}
$$

By simple analysis, we know that there exist some positive constants C_6, C_7, C_6' and C_7', such that

$$
C_6 \leq \int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{w}_k|^2dx \leq C_7 \quad \text{and} \quad C_6' \leq \int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{w}_k|^{p+1}dx \leq C_7'. \tag{3.33}
$$

Indeed, some calculations yields that

$$
\begin{gathered}
\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{u}_k|^2dx = \hat{\varepsilon}_k^{-2s}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{w}_k|^2dx, \\
\int_{\mathbb{R}^N}|\hat{u}_k|^{p+1}dx = \hat{\varepsilon}_k^{-\frac{N(p-1)}{2}}\int_{\mathbb{R}^N}|\hat{w}_k|^2dx.
\end{gathered} \tag{3.34}
$$

Applying (1.9), (3.31) and (3.34), we conclude that

$$
I_{M_k} = \hat{\varepsilon}_k^{-2s}\Big[\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{w}_k|^2dx + \hat{\varepsilon}_k^{2s}\frac{2M_k^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N}(1-m(x))|\tilde{u}_k|^{p+1}dx - \frac{2}{p+1}\int_{\mathbb{R}^N}|\hat{w}_k|^{p+1}dx\Big].
$$

This implies from Lemma 3.3 and Remark 3.5 that

$$
(a^*)^{\frac{2s(p-1)}{4s-N(p-1)}}\Big(\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{w}_k|^2dx - \frac{2}{p+1}\int_{\mathbb{R}^N}|\hat{w}_k|^{p+1}dx\Big) \to -\lambda < 0. \tag{3.35}
$$

If $\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{w}_k|^2dx \to \infty$ as $k\to\infty$. Setting $\gamma_k^2 := \int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{w}_k|^2dx$ and $\hat{w}_k(x) := \gamma_k^{N/2}u_k(\gamma_k x)$, one can deduce that $\|(-\Delta)^{s/2}u_k\|_2^2 = \|u_k\|_2^2 = 1$. Applying the Gagliardo-Nirenberg-Sobolev inequality (1.3), it yields $\|u_k\|_{p+1}^{p+1} \le C_{\rm opt}$. Moreover, we can deduce that

$$\frac{\int_{\mathbb{R}^N}|\hat{w}_k|^{p+1}dx}{\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{w}_k|^2dx} = \frac{\int_{\mathbb{R}^N}|\hat{u}_k|^{p+1}dx}{\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{u}_k|^2dx}\gamma_k^{\frac{N(p-1)-4s}{2}} \le C_{\rm opt}\gamma_k^{\frac{N(p-1)-4s}{2}} \to 0 \tag{3.36}$$

as $k\to\infty$. However, from (3.35), we have

$$\frac{\int_{\mathbb{R}^N}|\hat{w}_k|^{p+1}dx}{\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{w}_k|^2dx} \to 1, \quad \text{as } k\to\infty,$$

which contradicts (3.36), thus $\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{w}_k|^2dx \le C_7$. This fact combined with Gagliardo-Nirenberg-Sobolev inequality (1.3), we can conclude $\int_{\mathbb{R}^N}|\hat{w}_k|^{p+1}dx \le C_7'$. On the other hand, (3.35) shows that $\int_{\mathbb{R}^N}|\hat{w}_k|^{p+1}dx \ge C_6'$. Then (1.3) further implies that $\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\hat{w}_k|^2dx \ge C_6$.

Lemma 3.6. *Let u_k be a nonnegative minimizer of I_{M_k} as $k\to\infty$. Then there exist a sequence $\{y_k\}$, $R_0>0$ and $\eta>0$, such that the function*

$$w_k(x) := \hat{w}_k(x+y_k) = \hat{\varepsilon}_k^{N/2}u_k(\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k), \tag{3.37}$$

satisfies

$$\liminf_{k\to\infty}\int_{B_{R_0}(0)} w_k^{p+1}dx \ge \eta > 0, \tag{3.38}$$

where $\hat{\varepsilon}_k$ is given in (3.31). *Moreover, $\{\hat{\varepsilon}_k y_k\}$ is uniformly bounded as $k\to\infty$, and for any sequence $\{y_k\}$, there exists a subsequence, still denoted by $\{y_k\}$, such that $z_k := \hat{\varepsilon}_k y_k \overset{k}{\to} y_0$, for some point $y_0\in\mathbb{R}^N$ and y_0 is a global maximum point of $m(x)$, that is $y_0=0$.*

Proof. First, we claim that there exist a sequence $\{y_k\}\in\mathbb{R}^N$ and $R_0,\eta>0$ such that

$$\liminf_{k\to\infty}\int_{B_{R_0}(y_k)} \hat{w}_k^{p+1}dx \ge \eta > 0. \tag{3.39}$$

Suppose by contradiction that for any $R>0$, there exists a subsequence $\{\hat{w}_k\}$, such that

$$\lim_{k\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_{R_0}(y)} \hat{w}_k^{p+1}dx = 0.$$

By the fact that $1<p<1+\frac{4s}{N}$, we deduce from [12, Lemma 2.8] that $\hat{w}_k\to 0$ as $k\to\infty$ in $L^{p+1}(\mathbb{R}^N)$, which contradicts (3.33). Then (3.37) and (3.39) show that (3.38) holds.

Next, we show that $\{\hat{\varepsilon}_k y_k\}$ is bounded. By contradiction, suppose that $\{\hat{\varepsilon}_k y_k\}$ is unbounded. Then for any sequence $\{y_k\}$, there exists a subsequence $\{y_k\}$ such that $\lim_{k\to\infty}|\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k|\to\infty$ and $\lim_{k\to\infty} m(\hat{\varepsilon}_k x+\hat{\varepsilon}_k y_k) = m_\infty < 1$, where the constants $m_\infty\ge 0$ and $R_0>0$ are given in (A1) and (3.38), respectively. By Remark 3.5 and (3.37), we have

$$\frac{2M_k^{\frac{p-1}{2}}}{p+1}\hat{\varepsilon}_k^{-\frac{N(p-1)}{2}}\int_{\mathbb{R}^N}(1-m(\hat{\varepsilon}_k x+\hat{\varepsilon}_k y_k))|\hat{w}_k|^{p+1}dx \to 0 \quad \text{as } k\to\infty,$$

from which and (3.38), we obtain that

$$0 = \frac{2M_k^{\frac{p-1}{2}}}{p+1}\hat{\varepsilon}_k^{-\frac{N(p-1)}{2}}\int_{\mathbb{R}^N}(1-m(\hat{\varepsilon}_k x+\hat{\varepsilon}_k y_k))|\hat{w}_k|^{p+1}dx \ge \frac{2}{p+1}\hat{\varepsilon}_k^{-2s}(1-m_\infty)\eta \quad \text{as } k\to\infty,$$

which is a contradiction. Thus, $\{\hat{\varepsilon}_k y_k\}$ is uniformly bounded as $k\to\infty$. Moreover, $z_k := \hat{\varepsilon}_k y_k \overset{k}{\to} y_0$, for some point $y_0\in\mathbb{R}^N$. Finally, using (3.38) and Fatou's Lemma, we know that

$$\liminf_{k\to\infty}\int_{\mathbb{R}^N}(1-m(\hat{\varepsilon}_k x+\hat{\varepsilon}_k y_k))|\hat{w}_k|^{p+1}dx \ge (1-m(y_0))\int_{\mathbb{R}^N}\liminf_{k\to\infty}|\hat{w}_k|^{p+1}dx \ge (1-m(y_0))\eta,$$

which with Remark 3.5, we can conclude that $y_0=0$. By the assumption (A2) that 0 is the unique maximum point of $m(x)$. Then the proof is complete. $\square$

To obtain the limitting behavior of w_k, we first concentrate on the L^∞-estimate and decay estimate of w_k.

Lemma 3.7. *Let w_k be given by Lemma 3.6. Then there exists a constant $C>0$, such that*

$$\|w_k\|_\infty \le C \quad forallk \in \mathbb{N}. \tag{3.40}$$

Moreover,

$$w_k \to 0 \quad \text{as } |x| \to \infty, \text{ uniformly for large } k. \tag{3.41}$$

Proof. Since u_k is a nonnegative minimizer of I_{M_k}, we can derive that u_k satisfies the Euler-Lagrange equation

$$(-\Delta)^s u_k = \mu_k u_k + M_k^{\frac{p-1}{2}} m(x) u_k^p \quad \text{in } \mathbb{R}^N, \tag{3.42}$$

where $\mu_k \in \mathbb{R}$ is a suitable Lagrange multiplier and satisfies

$$\mu_k = \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_k|^2 dx - M_k^{\frac{p-1}{2}} \int_{\mathbb{R}^N} m(x)|u_k|^{p+1} dx = I_{M_k} - \frac{p-1}{p+1} M_k^{\frac{p-1}{2}} \int_{\mathbb{R}^N} m(x)|u_k|^{p+1} dx.$$

Then from (3.37), we have

$$(-\Delta)^s w_k = \mu_k \hat{\varepsilon}_k^{2s} w_k + m(\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k) w_k^p. \tag{3.43}$$

Combining (1.9), (3.37), (3.43), Lemma 3.3 and Lemma 3.4, it follows that

$$\begin{aligned} \hat{\varepsilon}_k^{2s} \mu_k &= \hat{\varepsilon}_k^{2s} \Big(I_{M_k} + \frac{p-1}{p+1} M_k^{\frac{p-1}{2}} \int_{\mathbb{R}^N} (1-m(x))|u_k|^{p+1} dx - \frac{p-1}{p+1} M_k^{\frac{p-1}{2}} \int_{\mathbb{R}^N} |u_k|^{p+1} dx \Big) \\ &= \hat{\varepsilon}_k^{2s} I_{M_k} - \frac{p-1}{p+1} \int_{\mathbb{R}^N} |w_k|^{p+1} dx \\ &= -\lambda (a^*)^{-\frac{2s(p-1)}{4s-N(p-1)}} - \frac{p-1}{p+1} \int_{\mathbb{R}^N} |w_k|^{p+1} dx < 0 \quad \text{as } k \to \infty. \end{aligned} \tag{3.44}$$

By (3.43) and (3.44), we have

$$(-\Delta)^s w_k \le w_k^p. \tag{3.45}$$

By a similar argument to the one in [3, Lemma 2.4] or [8, Proposition A.1], (3.40) holds.

Now we show $w_k \to 0$ as $|x| \to \infty$ uniformly for large k. The problem (3.43) can be rewritten as

$$(-\Delta)^s w_k = h_k(x) \quad \text{in } \mathbb{R}^N,$$

where

$$h_k(x) := \mu_k \hat{\varepsilon}_k^{2s} w_k + (m(\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k) - 1) w_k^p + w_k^p.$$

According to the argument in [8, Proposition A.1], we know that $w_k(x) \in L^r(\mathbb{R}^N)$ for $2 \le r < \infty$. Using (3.33) and (3.44), we can conclude that the uniform boundedness $\{\mu_k \hat{\varepsilon}_k^{2s}\}$ as $k \to \infty$. And by Lemma 3.6 that $\{\hat{\varepsilon}_k y_k\}$ is bounded in $\mathbb{R}^N$. As for the inhomogeneous term, from Remark 3.5 and (3.32), one can deduce that

$$\int_{\mathbb{R}^N} (m(\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k) - 1) w_k^{p+1} dx = \hat{\varepsilon}_k^{\frac{N(p-1)}{2}} \int_{\mathbb{R}^N} (m(x)-1) u_k^{p+1} dx \to 0 \quad \text{as } k \to \infty. \tag{3.46}$$

Combining the above facts with (3.40), we deduce that $h_k \in C^\infty(\mathbb{R}^N)$ for large k. Thus, using (3.40) and [32, Proposition 2.9], we have

$$\|w_k\|_{C^{1,\alpha}(\mathbb{R}^N)} \le C(\|w_k\|_\infty + \|h_k\|_\infty) \le C \quad \text{for } \alpha < 2s-1, \text{ as } k \to \infty.$$

Finally, the fact $w_k \in L^r(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ for $2 \le r < \infty$, which implies that $\lim_{|x|\to\infty} w_k(x) = 0$ uniformly for large k. Thus, (3.41) holds. □

Lemma 3.8. *Let w_k be given by Lemma 3.6. Then there exists a constant $C>0$, such that*

$$\|w_k\|_\infty \le \frac{C}{1+|x|^{N+2s}} \quad \text{uniformly for large } k,$$

where $C>0$ is a constant independent of k.

Proof. By the boundedness of $\{\mu_k \hat{\varepsilon}_k^{2s}\}$, we may assume that

$$\mu_k \hat{\varepsilon}_k^{2s} \to -\gamma, \tag{3.47}$$

for some $\gamma \in \mathbb{R}^+$, as $k \to \infty$. According [32, Lemma 4.3], there exists a function ψ, such that

$$0 \le \psi \le \frac{C}{1+|x|^{N+2s}}, \tag{3.48}$$

$$(-\Delta)^s \psi + \frac{\gamma}{2}\psi = 0 \quad \text{in } \mathbb{R}^N \backslash B_{R_1(0)}, \tag{3.49}$$

for some suitable $R_1 > 0$. By (3.47), we deduce that there exists $R_2 > 0$ sufficiently large such that for large k,

$$(-\Delta)^s w_k + \frac{\gamma}{2} w_k \le \mu_k \hat{\varepsilon}_k^{2s} w_k + m(\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k) w_k^p + \frac{\gamma}{2} w_k \le 0 \quad \text{for } |x| \ge R_2. \tag{3.50}$$

Let $R_3 = \max\{R_1, R_2\}$. For large k, set

$$d := \inf_{B_{R_3(0)}} \psi > 0 \quad \text{and} \quad \breve{w}_k = (\delta+1)\psi - d w_k, \tag{3.51}$$

where $\delta = \sup \|w_k\|_\infty < \infty$, we claim that $\breve{w}_k \ge 0$ uniformly for large k. Indeed, if not, then there exists a sequence $\{x_j\}$ such that

$$\inf_{x\in\mathbb{R}^N} \breve{w}_k(x) = \lim_{j\to\infty} \breve{w}_k(x_j) < 0. \tag{3.52}$$

Combining (3.41) and (3.48), we deduce that

$$\lim_{|x|\to\infty} \breve{w}_k(x) = 0 \quad \text{uniformly for large } k. \tag{3.53}$$

Then, (3.52) and (3.53) show that $\{x_j\}$ is bounded. Moreover, up to a subsequence, let $x_j \to x_*$ for some $x_* \in \mathbb{R}^N$ as $j \to \infty$. From (3.52), we have

$$\inf_{x\in\mathbb{R}^N} \breve{w}_k(x) = \breve{w}_k(x_*) < 0, \tag{3.54}$$

which implies that

$$(-\Delta)^s \breve{w}_k(x_*) = -\frac{C_s}{2}\int_{\mathbb{R}^N} \frac{\breve{w}_k(x_*+y) + \breve{w}_k(x_*-y) - 2\breve{w}_k(x_*)}{|y|^{N+2s}} dy \le 0, \tag{3.55}$$

from which and (3.51), we obtain $\breve{w}_k(x_*) = \delta\psi + \psi - d w_k \ge \delta d + \psi - d\delta > 0$ in $B_{R_3(0)}$. Thus, (3.54) shows that

$$x_* \in \mathbb{R}^N \backslash B_{R_3(0)}. \tag{3.56}$$

It follows from (3.48)-(3.51) that

$$(-\Delta)^s \breve{w}_k + \frac{\gamma}{2}\breve{w}_k \ge 0 \in \mathbb{R}^N \backslash B_{R_3(0)}. \tag{3.57}$$

Using (3.54)-(3.57), we deduce that $0 \le (-\Delta)^s \breve{w}_k(x_*) + \frac{\gamma}{2}\breve{w}_k(x_*) < 0$, which is a contradiction, so $\breve{w}_k \ge 0$ uniformly for large k. Then (3.47) shows that

$$w_k \le (\delta+1) d^{-1}\psi \le \frac{C}{1+|x|^{N+2s}} \quad \text{uniformly for large } k,$$

thus, we complete the proof of Lemma 3.8. □

The next Lemma shows the limitting behavior of w_k.

Lemma 3.9. *Let w_k be given by Lemma 3.6. Then passing to a subsequence if necessary, we have*

$$\lim_{k\to\infty} w_k(x) = (a^*)^{-\frac{2s}{4s-N(p-1)}} Q[(a^*)^{-\frac{p-1}{4s-N(p-1)}} x + \tilde{x}_0] \quad \text{in } H^s(\mathbb{R}^N) \text{ for some } \tilde{x}_0 \in \mathbb{R}^N,$$

where $a^* = \|Q\|_2^2$ and $Q(x)$ is the unique radially symmetric positive solution of (1.5).

Proof. From (3.33), we can know that w_k is bounded uniformly in $H^s(\mathbb{R}^N)$. Using (3.46) and taking $k \to \infty$, passing to a subsequence, then we have $w_k \rightharpoonup w_0$ in $H^s(\mathbb{R}^N)$ for some $w_0 \in H^s(\mathbb{R}^N)$ satisfies

$$(-\Delta)^s w_0 = -\gamma w_0 + w_0^p \quad \text{in } \mathbb{R}^N, \tag{3.58}$$

where $\gamma > 0$ be given by Lemma 3.8. By (3.38), we know that $w_0 \not\equiv 0$. Similar to the proof of [34, Proposition 4.4], we have that $w_0 \in C^{1,\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Applying [29, Lemma 3.2], we have

$$(-\Delta)^s \check{w}_0(x) = -\frac{C_s}{2} \int_{\mathbb{R}^N} \frac{\check{w}_0(x+y) + \check{w}_0(x-y) - 2\check{w}_0(x)}{|y|^{N+2s}} dy \quad \text{in } \mathbb{R}^N.$$

Assume that there exists $\bar{x} \in \mathbb{R}^N$, such that $w_0(\bar{x}) = 0$, this together with $w_0 \geq 0$ and $w_0 \not\equiv 0$ leads to

$$(-\Delta)^s \check{w}_0(\bar{x}) = -\frac{C_s}{2} \int_{\mathbb{R}^N} \frac{\check{w}_0(\bar{x}+y) + \check{w}_0(\bar{x}-y)}{|y|^{N+2s}} dy < 0.$$

However, it is evident that

$$(-\Delta)^s w_0(\bar{x}) = -\gamma w_0(\bar{x}) + w_0^p(\bar{x}) = 0,$$

which is a contradiction. Thus, $w_0(x) > 0$ for $x \in \mathbb{R}^N$. Since the equation (1.5), up to translations, admits a unique positive solution Q, it then follow from (3.58) that, there exists $\tilde{x}_0 \in \mathbb{R}^N$ such that

$$w_0(x) = \gamma^{\frac{1}{p-1}} Q(\gamma^{\frac{1}{2s}} x + \tilde{x}_0) \quad \text{for} \quad \text{some } \tilde{x}_0 \in \mathbb{R}^N. \tag{3.59}$$

We now claim that

$$\int_{\mathbb{R}^N} m(\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k) w_k^{p+1}(x) dx \to \int_{\mathbb{R}^N} m(\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k) w_0^{p+1}(x) dx. \tag{3.60}$$

Noting that

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^N} m(\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k) w_k^{p+1}(x) dx - \int_{\mathbb{R}^N} m(\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k) w_0^{p+1}(x) dx \Big| \\
&\leq \int_{\mathbb{R}^N} |w_k^{p+1}(x) - w_0^{p+1}(x)| dx \\
&= \int_{B_R(0)} |w_k^{p+1}(x) - w_0^{p+1}(x)| dx + \int_{\mathbb{R}^N \setminus B_R(0)} |w_k^{p+1}(x) - w_0^{p+1}(x)| dx \\
&:= A_k + B_k
\end{aligned}$$

where $R > 0$ is arbitrary. By the fact that $|a^m - b^m| \leq L|a-b|^m$ for $a, b \geq 0, m \geq 1$ and $L \geq 1$, we have

$$A_k \leq L \int_{B_R(0)} |w_k^{p+1}(x) - w_0^{p+1}(x)| dx \to 0 \quad \text{as } k \to \infty, \tag{3.61}$$

due to Sobolev embedding theorem and the uniform boundedness of w_k in $\mathbb{R}^N$. On the other hand, from Lemma 3.8, (1.4) and (3.59), for any $\varepsilon > 0$, there exists a constant $R_\varepsilon > 0$ independent of k, such that

$$\begin{aligned}
B_k &\leq \int_{\mathbb{R}^N \setminus B_R(0)} |w_k^{p+1}(x) + w_0^{p+1}(x)| dx \\
&\leq \int_{\mathbb{R}^N \setminus B_R(0)} \frac{C}{|x|^{(N+2s)(p+1)}} dx \\
&\leq C R^{N-(N+2s)(p+1)} \leq C\varepsilon \quad \forall R > R_\varepsilon \text{ as } k \to \infty,
\end{aligned} \tag{3.62}$$

since the arbitrariness of ε, we conclude from (3.61) and (3.62) that (3.60) holds.

Next, we prove that $\|w_0\|_2^2 = 1$. By contradiction, we assume that $\|w_0\|_2^2 = l$, where $l \in (0,1)$, due to $\|w_0\|_2^2 \leq \lim_{k\to\infty} \|w_k\|_2^2 = 1$. Using the Brézis-Lieb Lemma, we have

$$\begin{gathered}
\|w_k\|_2^2 = \|w_0\|_2^2 + \|w_k - w_0\|_2^2 + o(1) \quad \text{as } k \to \infty, \\
\|w_k\|_{p+1}^{p+1} = \|w_0\|_{p+1}^{p+1} + \|w_k - w_0\|_{p+1}^{p+1} + o(1) \quad \text{as } k \to \infty, \\
\|(-\Delta)^{s/2} w_k\|_2^2 = \|(-\Delta)^{s/2} w_0\|_2^2 + \|(-\Delta)^{s/2}(w_k - w_0)\|_2^2 + o(1) \quad \text{as } k \to \infty.
\end{gathered} \tag{3.63}$$

Set $w_l := \frac{w_0}{l}$, it follows from (3.8), (3.34)-(3.37), (3.60), (3.63) and Lemma 3.3 that

$$\begin{aligned}
\tilde{I}_1 &= \lim_{k\to\infty} \hat{\varepsilon}_k^{2s}\Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_k|^2dx - \frac{2M_k^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^N} m(x)|u_k|^{p+1}dx\Big)\\
&= \lim_{k\to\infty}\Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}w_k|^2dx - \frac{2}{p+1}\int_{\mathbb{R}^N} m(\hat{\varepsilon}_k x+\hat{\varepsilon}_k y_k)|w_0|^{p+1}dx\Big)\\
&\geq \lim_{k\to\infty}\Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}w_k|^2dx - \frac{2}{p+1}\int_{\mathbb{R}^N} |w_0|^{p+1}dx\Big)\\
&= \int_{\mathbb{R}^N} |(-\Delta)^{s/2}w_0|^2dx + \lim_{k\to\infty}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}(w_k-w_0)|^2dx - \frac{2}{p+1}\int_{\mathbb{R}^N} |w_0|^{p+1}dx\\
&\geq \int_{\mathbb{R}^N} |(-\Delta)^{s/2}w_0|^2dx - \frac{2}{p+1}\int_{\mathbb{R}^N} |w_0|^{p+1}dx\\
&= l\int_{\mathbb{R}^N} |(-\Delta)^{s/2}w_l|^2dx - \frac{l^{\frac{p+1}{2}}}{p+1}\int_{\mathbb{R}^N} |w_l|^{p+1}dx\\
&= l\Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}w_l|^2dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |w_l|^{p+1}dx\Big) \geq l\tilde{I}_1,
\end{aligned}$$

which implies that $l>1$, due to $\tilde{I}_1<0$, this is a contradiction. Thus, $\|w_0\|_2^2 = 1$. Moreover, we can obtain that $\gamma = \|Q\|_2^{-\frac{4s(p-1)}{4s-N(p-1)}}$. Since $\|w_k\|_2^2 = \|w_0\|_2^2 = 1$, one can derive that $w_k \to w_0$ in $L^2(\mathbb{R}^N)$ as $k\to\infty$. Then using the interpolation inequality and the Sobolev inequality, we have

$$w_k \to w_0 \quad \text{in } L^q(\mathbb{R}^N),\ q\in[2,2_s^*) \text{ as } k\to\infty.$$

Further, substituting $\gamma = \|Q\|_2^{-\frac{4s(p-1)}{4s-N(p-1)}}$ into (3.59), we can deduce that Lemma 3.9 holds. □

3.3. **Proof of main resutls.** In this section, we prove Theorem 1.3 on the concentration behavior of minimizers for I_M with $M\to\infty$. We first give the proof of Theorem 3.1.

Proof of Theorem 3.1. From (3.41), one knows that u_k admits at least one global maximum point. Let $\bar{z}_k$ be any global maximum point of u_k, according to Lemma 3.6 that $z_k := \hat{\varepsilon}_k y_k \to 0$ as $k\to\infty$. Hence, we deduce from (3.37) that $w_k(x)$ attains its global maximum point at $x_k = \frac{\bar{z}_k - z_k}{\hat{\varepsilon}_k}$. One can verify that $\{\frac{\bar{z}_k - z_k}{\hat{\varepsilon}_k}\}$ is bounded uniformly in $\mathbb{R}^N$. Otherwise, it follows from (3.41) that $\lim_{k\to\infty}\|w_k\|_\infty = 0$ as $|x_k|\to\infty$, which contradicts to (3.38). This further implies that, passing to a subsequence if necessary

$$\lim_{k\to\infty} \bar{z}_k = \lim_{k\to\infty} z_k = 0. \tag{3.64}$$

By Lemma 3.9, we have

$$w_k(x) \to (a^*)^{-\frac{2s}{4s-N(p-1)}} Q\Big((a^*)^{-\frac{p-1}{4s-N(p-1)}}x + \tilde{x}_0\Big) \tag{3.65}$$

in $H^s(\mathbb{R}^N)$ for some $\tilde{x}_0\in\mathbb{R}^N$, as $k\to\infty$, where w_k is given by Lemma 3.6. We next prove that

$$w_k(x) \to (a^*)^{-\frac{2s}{4s-N(p-1)}} Q\Big((a^*)^{-\frac{p-1}{4s-N(p-1)}}x + \tilde{x}_0\Big) \quad \text{in } C^2_{\rm loc}(\mathbb{R}^N) \text{ as } k\to\infty. \tag{3.66}$$

Indeed, it follows from (3.37) that

$$(-\Delta)^s w_k = \mu_k\hat{\varepsilon}_k^{2s}w_k + m(\hat{\varepsilon}_k x+\hat{\varepsilon}_k y_k)w_k^p := g(w_k) \quad \text{in } \mathbb{R}^N, \tag{3.67}$$

where $\mu_k\in\mathbb{R}$ is a suitable Lagrange multiplier. Similar to the proof of Lemma 3.6, we know that $\|w_k\|_\infty \le C$ uniformly in k, then (3.64) shows that $g(w_k)\in L^\infty(\mathbb{R}^N)$. Applying [32, Proposition 2.9], we have

$$\|w_k\|_{C^{1,\alpha}(\mathbb{R}^N)} \le C(\|w_k\|_\infty + \|g(w_k)\|_\infty) \le C \quad \text{for some } \alpha\in(0,1) \text{ as } k\to\infty. \tag{3.68}$$

From this, (A1) and $\|w_k\|_\infty \le C$, we assert that

$$\|m(\hat{\varepsilon}_k x+\hat{\varepsilon}_k y_k)w_k^p\|_{C^{0,\alpha}(\mathbb{R}^N)}$$

$$
\begin{aligned}
&\le C+\sup_{x\neq y}\frac{|m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)-m(\hat\varepsilon_k y+\hat\varepsilon_k y_k)||w_k^p(x)|}{|x-y|^\alpha}+\sup_{x\neq y}\frac{|m(\hat\varepsilon_k y+\hat\varepsilon_k y_k)||w_k^p(x)-w_k^p(y)|}{|x-y|^\alpha}\\
&\le C+C_8\sup_{x\neq y}\frac{|m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)-m(\hat\varepsilon_k y+\hat\varepsilon_k y_k)|}{|x-y|^\alpha}\\
&\quad+\sup_{x\neq y}\frac{|m(\hat\varepsilon_k y+\hat\varepsilon_k y_k)||w_k(x)-w_k(y)|}{|x-y|^\alpha}|w_k^{p-1}(x)+w_k^{p-1}(y)|\\
&\le C+C_8\sup_{x\neq y}\frac{|m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)-m(\hat\varepsilon_k y+\hat\varepsilon_k y_k)|}{|x-y|^\alpha}+C_9\sup_{x\neq y}\frac{|w_k(x)-w_k(y)|}{|x-y|^\alpha},
\end{aligned}
$$

for any $t>0$ fixed, we obtain

$$
\begin{aligned}
&\frac{|w_k(x)-w_k(y)|}{|x-y|^\alpha}\le CT^{1-\alpha}\le C\quad\text{if } |x-y|\le T,\\
&\frac{|w_k(x)-w_k(y)|}{|x-y|^\alpha}\le CT^{-\alpha}\le C\quad\text{if } |x-y|> T,
\end{aligned}
\tag{3.69}
$$

and

$$
\begin{aligned}
&\frac{|m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)-m(\hat\varepsilon_k y+\hat\varepsilon_k y_k)|}{|x-y|^\alpha}\le CT^{1-\alpha}\le C\quad\text{if } |x-y|\le T,\\
&\frac{|m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)-m(\hat\varepsilon_k y+\hat\varepsilon_k y_k)|}{|x-y|^\alpha}\le CT^{-\alpha}\le C\quad\text{if } |x-y|> T.
\end{aligned}
\tag{3.70}
$$

It follows from (3.69) and (3.70) that

$$
\|m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)w_k^p\|_{C^{0,\alpha}(\mathbb{R}^N)}\le C\quad\text{for large } k.\tag{3.71}
$$

The same procedure may be easily adapted to obtain

$$
\|w_k\|_{C^{0,\alpha}(\mathbb{R}^N)}\le C\quad\text{for large } k.\tag{3.72}
$$

Hence, combining (3.71) and (3.72), we conclude $\|g(w_k)\|_{C^{0,\alpha}(\mathbb{R}^N)}\le C$ for large k. Applying [32, Proposition 2.8], we obtain

$$
\|w_k\|_{C^{1,\varrho_1}(\mathbb{R}^N)}\le C(\|w_k\|_\infty+\|g(w_k)\|_{C^{0,\alpha}(\mathbb{R}^N)})\le C\quad\text{for large } k,\tag{3.73}
$$

where $\varrho_1=\alpha+2s-1$. Differentiating (3.67), we obtain

$$
(-\Delta)^s(w_k)_{x_i}=\mu_k\hat\varepsilon_k^{2s}(w_k)_{x_i}+\hat\varepsilon_k m'(\hat\varepsilon_k x+\hat\varepsilon_k y_k)w_k^p+m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)pw_k^{p-1}(w_k)_{x_i}:=G(x)
$$

for $i=1,2,\dots,N$. Obviously, (3.73) shows that

$$
\|(w_k)_{x_i}\|_{C^{0,\varrho_1}(\mathbb{R}^N)}\le C\quad\text{and}\quad \|m'(\hat\varepsilon_k x+\hat\varepsilon_k y_k)w_k^p\|_{C^{0,\varrho_1}(\mathbb{R}^N)}\le C\quad\text{for large } k.\tag{3.74}
$$

Using (3.73), (3.74) and (A1), we conclude that

$$
\begin{aligned}
&\|m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)w_k^{p-1}(w_k)_{x_i}\|_{C^{0,\alpha}(\mathbb{R}^N)}\\
&\le C+\sup_{x\neq y}\frac{\Big|m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)w_k^{p-1}(x)\nabla w_k(x)-m(\hat\varepsilon_k y+\hat\varepsilon_k y_k)w_k^{p-1}(y)\nabla w_k(y)\Big|}{|x-y|^\alpha}\\
&\le C+\sup_{x\neq y}\frac{m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)|w_k^{p-1}(x)-w_k^{p-1}(y)||\nabla w_k(x)|}{|x-y|^\alpha}\\
&\quad+\sup_{x\neq y}\frac{m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)|w_k^{p-1}(y)||\nabla w_k(x)-\nabla w_k(y)|}{|x-y|^\alpha}\\
&\quad+\sup_{x\neq y}\frac{|w_k^{p-1}(y)||\nabla w_k(y)||m(\hat\varepsilon_k x+\hat\varepsilon_k y_k)-m(\hat\varepsilon_k y+\hat\varepsilon_k y_k)|}{|x-y|^\alpha}\le C\quad\text{for large } k.
\end{aligned}
\tag{3.75}
$$

We also conclude that

$$
\|(w_k)_{x_i}\|_{C^{0,\alpha}(\mathbb{R}^N)}\le C\quad\text{and}\quad \|m'(\hat\varepsilon_k x+\hat\varepsilon_k y_k)w_k^p\|_{C^{0,\alpha}(\mathbb{R}^N)}\le C\quad\text{for large } k,\tag{3.76}
$$

in the same way. By the fact that $\|w_k\|_\infty \le C$ uniformly in k, we then derive from (3.75) and (3.76) that $\|G(x)\|_\infty \le C$. Applying [32, Proposition 2.8] again, we deduce that for some $\varrho_2 \in (0,1)$,

$$\|(w_k)_{x_i}\|_{C^{1,\varrho_2}(\mathbb{R}^N)} \le C(\|(w_k)_{x_i}\|_\infty + \|G(x)\|_\infty) \le C \quad \text{for large } k, \tag{3.77}$$

which implies that $\|w_k\|_{C^2(\mathbb{R}^N)} \le C$. Thus, (3.66) holds.

Noting that the origin is a local maximum point of w_k for all $k > 0$, and it follows from (3.66) that it is also a global maximum point of Q. Moreover, it is evident that $x = 0$ is the unique global maximum point of Q, then we conclude that $\tilde{x}_0 = 0$. Thus, we have

$$w_k \to (a^*)^{-\frac{2s}{4s-N(p-1)}} Q\left((a^*)^{-\frac{p-1}{4s-N(p-1)}} x\right) \quad \text{in } H^s(\mathbb{R}^N) \text{ as } k \to \infty. \tag{3.78}$$

Finally, we shall prove the uniqueness of global maximum point of w_k as k is large enough. From the definition of $\bar{z}_k$ and (3.67), we have $(-\Delta)^s w_k(\bar{z}_k) \ge C > 0$. Using Remark 3.5, (3.47) and (3.67), one can deduce that$w_x(\bar{z}_k) \ge C > 0$ for large k. It then follows from (3.41) that all local maximum points of w_k must stay in a finite ball $B_{\bar{R}(0)}$ in $\mathbb{R}^N$ as k is large enough, where $\bar{R}$ is independent of k. Letting $\bar{R} > 0$ small enough, such that $Q''(\iota) < 0$ for $0 < \iota < \bar{R}$. Then [30, Lemma 4.2] implies that $\{w_k\}$ has no critical points other than the origin, that is, x_k is the unique maximum point of u_k as k is large enough. Hence, we complete the proof of Theorem 3.1. □

Proof of the Theorem 1.3. Now, based on the Theorem 3.1, let v_k be a nonnegative minimizer of $I(M_k)$ and $\varepsilon_k := \left(\frac{M_k}{a^*}\right)^{-\frac{p-1}{4s-N(p-1)}}$. Note from Theorem 3.1 that $\varepsilon_k = (a^*)^{\frac{p-1}{4s-N(p-1)}} \hat{\varepsilon}_k$ and $v_k = M_k^{1/2} u_k$, it follows that

$$\begin{aligned}
\varepsilon_k^{\frac{2s}{p-1}} v_k(\varepsilon_k x + \bar{z}_k) &= (a^*)^{\frac{2s}{4s-N(p-1)}} \hat{\varepsilon}_k^{\frac{2s}{p-1}} M_k^{1/2} u_k(\varepsilon_k x + \bar{z}_k) \\
&= (a^*)^{\frac{2s}{4s-N(p-1)}} \hat{\varepsilon}_k^{N/2} u_k(\varepsilon_k x + \bar{z}_k) \\
&= (a^*)^{\frac{2s}{4s-N(p-1)}} (a^*)^{-\frac{2s}{4s-N(p-1)}} Q\Big((a^*)^{-\frac{p-1}{4s-N(p-1)}} \frac{\varepsilon_k}{\hat{\varepsilon}_k} x\Big) \\
&= Q(x) \quad \text{in } H^s(\mathbb{R}^N) \text{ as } k \to \infty,
\end{aligned} \tag{3.79}$$

which implies that (1.8) holds. Moreover, as for the uniqueness and limit behavior of the local maximum point $\bar{z}_k$ of v_k, we can obtain from Theorem 3.1 directly. Consequently, we complete the proof. □

References

[1] S. Y. A. Chang, M. M. González; *Fractional Laplacian in conformal geometry.* Adv. Math. 226 (2011) ,1410-1432.

[2] M. Cheng; *Bound state for the fractional Schrödinger equation with unbounded potential.* J. Math. Phys. 53 (2012), 043507.

[3] S. Cingolani, M. Gallo, K. Tanaka; *On fractional Schrödinger equations with Hartree type nonlinearities.* Math. Eng. 4 (2022), 1-33.

[4] R. Cont, P. Tankov; *Financial modelling with jump processes.* Chapman and Hall/CRC Financial Mathematics Series. Boca Raton, 2003.

[5] F.Y. Deren, T. S. Cerdik, R. P. Agarwal; *Existence criteria of positive solutions for fractional p-Laplacian boundary value problems.* Filomat 34 (2020), 3789-3799.

[6] Y. B. Deng, Y. J. Guo, L. Lu; *On the collapse and concentration of Bose-Einstein condensates with inhomogeneous attractive interactions.* Calc. Var. Partial Differential Equations. 54 (2015), 99-118.

[7] Y. B. Deng, Y. J. Guo, L. Lu; *Threshold behavior and uniqueness of ground states for mass critical inhomogeneous Schrödinger equations.* J. Math. Phys. 59 (2018), 011503.

[8] Y. B. Deng, S. J. Peng, X. Yang; *Existence and decays of solutions for fractional Schrödinger equations with general potentials.* Calc. Var. Partial Differential Equations. 63 (2024), 128.

[9] S. Dipierro, G. Palatucci, E. Valdinoci; *Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian.* Matematiche 68 (2013), 201-216.

[10] M. Du, L. X. Tian, J. Wang, F. B. Zhang; *Existence of normalized solutions for nonlinear fractional Schrödinger equations with trapping potentials.* Proc. Roy. Soc. Edinburgh Sect. A. 149 (2019), 617-653.

[11] P. Felmer, A. Quaas, J. Tan; *Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian.* Proc. Roy. Soc. Edinburgh Sect. A, 142 (2012), 1237-1262.

[12] B. H. Feng; *Ground states for the fractional Schrödinger equation.* Electron. J. Differential Equations. 2013 (2013) No. 127, 1-11.

[13] Y. S. Gao, S. Li; *Constraint minimizers of inhomogeneous mass subcritical minimization problems*. Math. Methods Appl. Sci. 44 (2021), 10062-10075.

[14] Y. J. Guo, C. S. Lin, J. C. Wei; *Local uniqueness and refined spike profiles of ground states for two-dimensional attractive Bose-Einstein condensates*. SIAM J. Math. Anal. 49,3671-3715 (2017).

[15] Y. J. Guo, R. Seiringer; *On the mass concentration for Bose-Einstein condensates with attractive interactions*. Lett. Math. Phys. 104 (2014), 141-156.

[16] Y. J. Guo, Z. Q. Wang, X. Y. Zeng, H. S. Zhou; *Properties of ground states of attractive Gross-Pitaevskii equations with multi-well potentials*. Nonlinearity. 31 (2018), 957-979.

[17] Y. J. Guo, X. Y. Zeng, H. S. Zhou; *Energy estimates and symmetry breaking in attractive Bose-Einstein condensates with ring-shaped potentials*. Ann. Inst. H. Poincaré C Anal. Non Linéaire. 33 (2016), 809-828.

[18] Y. J. Guo, X. Y. Zeng, H. S. Zhou; *Concentration behavior of standing waves for almost mass critical nonlinear Schrödinger equations*. J. Differential Equations. 256 (2019), 2079-2100.

[19] N. Laskin; *Fractional quantum mechanics and Lévy path integrals*. Phys. Lett. A, 268 (2000) no.4-6, 298-305.

[20] N. Laskin; *Fractional Schrödinger equation*. Phys. Rev. E., 66 (2022), 056108.

[21] P. L. Lions; *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire., 1 (1984), 109-145.

[22] P. L. Lions; *The concentration-compactness principle in the calculus of variations*. The locally compact case. II, Ann. Inst. H. Poincaré Anal. Non Linéaire. 1 (1984), 223-283.

[23] L. T. Liu, K. M. Teng, J. Yang, H. B. Chen; *Properties of minimizers for the fractional Kirchhoff energy functional*. J. Math. Phys. 64 (2023) No. 8, 081504.

[24] G. B. Li, H. Y. Y. Hongyu; *On the concentration phenomenon of L^2-subcritical constrained minimizers for a class of Kirchhoff equations with potentials*. J. Differential Equations. 266 (2019), 7101-7123.

[25] S. Li, X. C. Zhu; *Mass concentration and local uniqueness of ground states for L^2-subcritical nonlinear Schrödinger equations*. Z. Angew. Math. Phys. 70 (2016), 1-26.

[26] S. Li, Q. Zhang, X. C. Zhu; *Existence and limit behavior of prescribed L^2-norm solutions for Schrödinger-Poisson-Slater systems in $\mathbb{R}^3$*. Math. Methods Appl. Sci. 40 (2017), 7705-7721.

[27] M. Maeda; *On the symmetry of the ground states of nonlinear Schrödinger equation with potential*. Adv. Nonlinear Stud. 10 (2010), 895-925.

[28] R. Metzler, J. Klafter; *The random walk's guide to anomalous diffusion: a fractional dynamics approach*. Phys. Rep. 339 (2000), 1-77.

[29] E. D. Nezza, G. Palatucci, E. Valdinoci; *Hitchhiker's guide to the fractional Sobolev spaces*. Bull. Sci. Math. 136 (2012), 521-573.

[30] W. M. Ni, I. Takagi; *On the shape of least-energy solutions to a semilinear Neumann problem*. Comm. Pure Appl. Math. 44 (1991), 819-851.

[31] M. A. Ragusa; *Parabolic Herz spaces and their applications*. Appl. Math. Lett. 25 (2012), 1270-1273.

[32] L. Silvestre; *Regularity of the obstacle problem for a fractional power of the Laplace operator*. Comm. Pure Appl. Math. 60 (2007), 67-112.

[33] K. M. Teng; *Existence of ground state solutions for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent*. J. Differential Equations. 261 (2016), 3061-3106.

[34] K. M. Teng, R. P. Agarwal; *Existence and concentration of positive ground state solutions for nonlinear fractional Schrödinger-Poisson system with critical growth*. Math. Methods Appl. Sci. 42 (2018), 8258-8293.

[35] K. M. Teng, Y. Q. Cheng; *Multiplicity and concentration of nontrivial solutions for fractional Schrödinger-Poisson system involving critical growth*. Nonlinear Anal. 202 (2021), 112144.

[36] X. F. Wang; *On concentration of positive bound states of nonlinear Schrödinger equations*. Comm. Math. Phys., 153 (1993), 229-244.

[37] Y. Wu, S. Taarabti; *Existence of two positive solutions for two kinds of fractional p-Laplacian equations*. J. Funct. Spaces. 1 (2021), 5572645.

[38] J. Yang, H. B. Chen, L. T. Liu; *Limiting behaviors of constrained minimizers for the mass subcritical fractional NLS equations*. Anal. Math. Phys. 14 (2024), 1-28.

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